Two Enumerative Tidbits

Richard P. Stanley

M.I.T.

The first tidbit

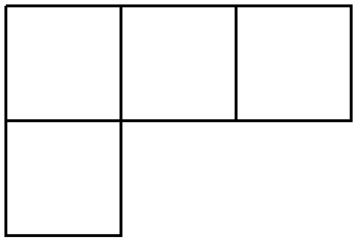
The Smith normal form

of some matrices

connected with Young diagrams

Extended Young diagrams

$\boldsymbol{\lambda}$: a partition $(\lambda_1, \lambda_2, \ldots)$, identified with its Young diagram



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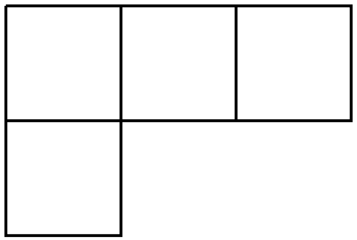

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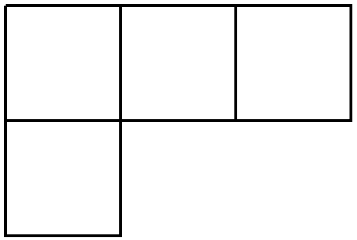

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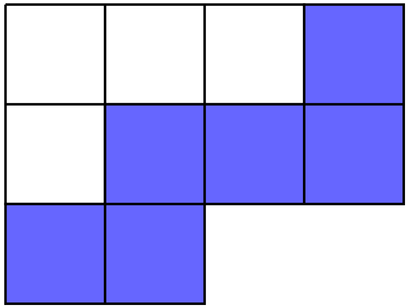

(3,1)* = (4,4,2)

Initialization

Insert 1 into each square of λ^*/λ.

			1
	1	1	1
1	1		

$(3,1)^* = (4,4,2)$

$\boldsymbol{M_t}$

Let $t \in \lambda$. Let $\boldsymbol{M_t}$ be the largest square of λ^* with t as the upper left-hand corner.

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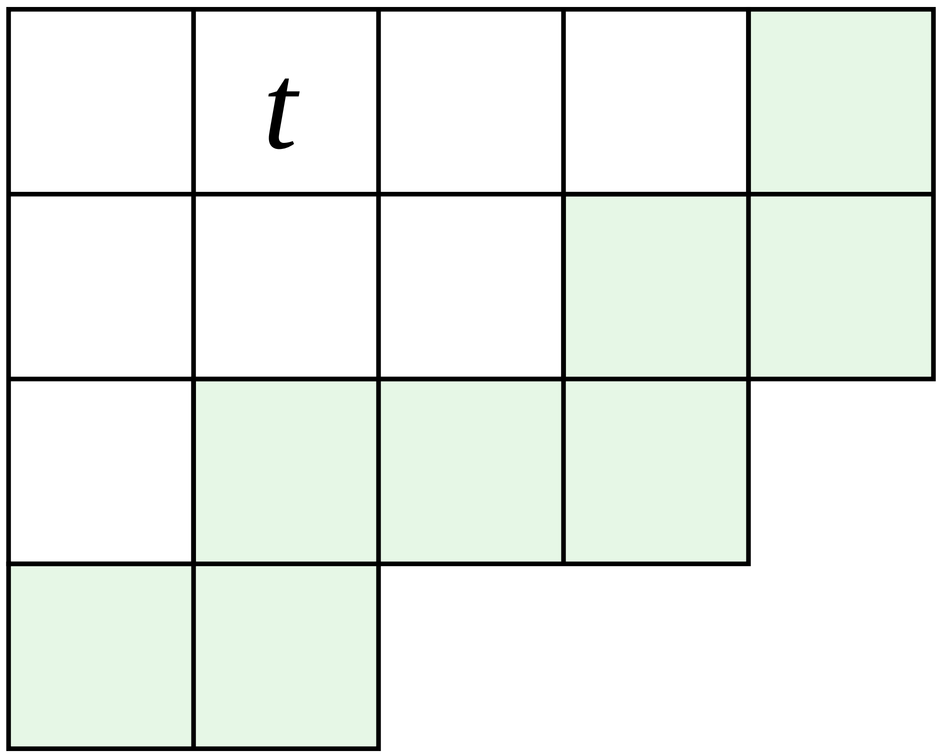

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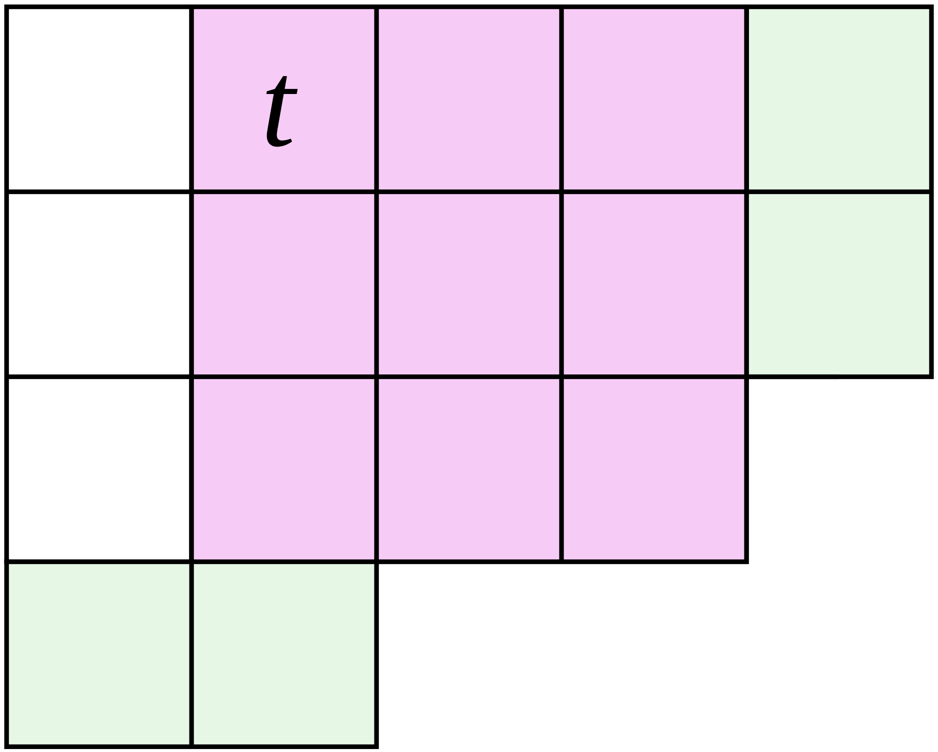

Determinantal algorithm

Suppose all squares to the southeast of t have been filled. Insert into t the number $\boldsymbol{n_t}$ so that $\det M_t = 1$.

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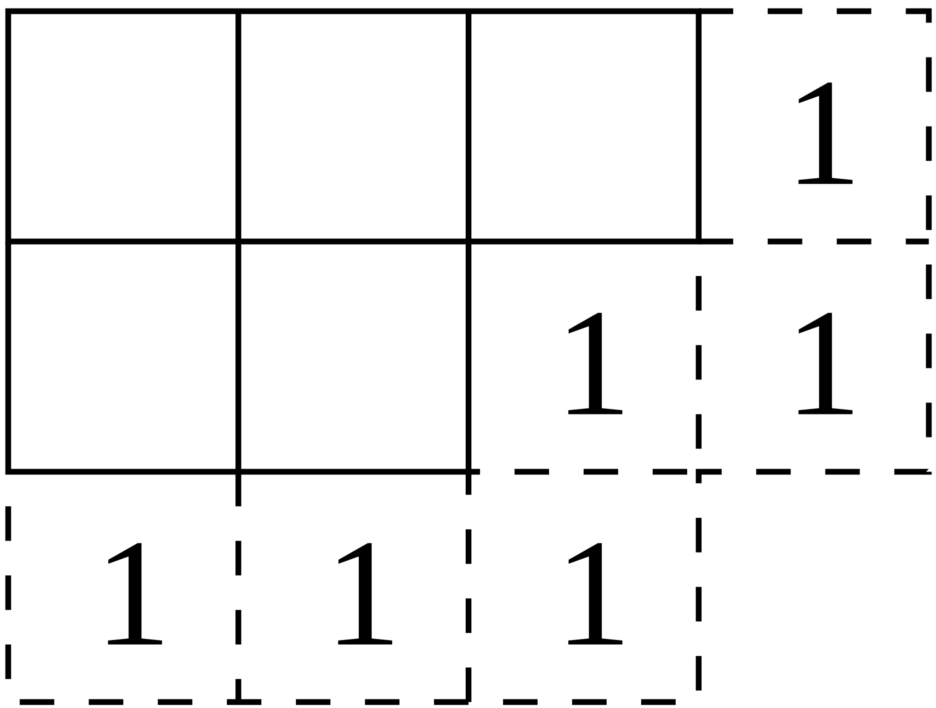

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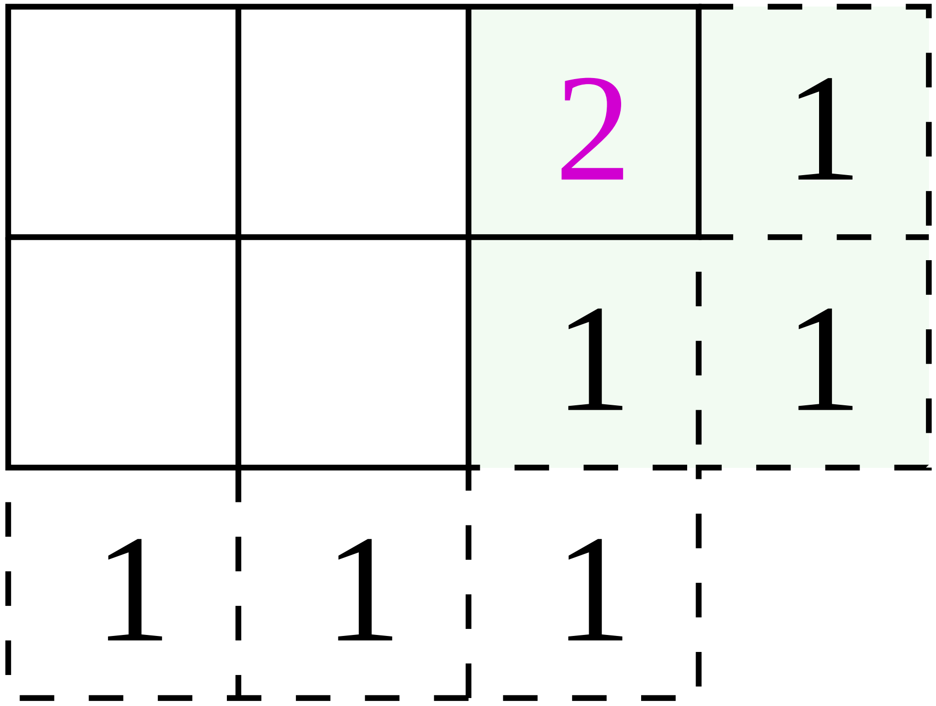

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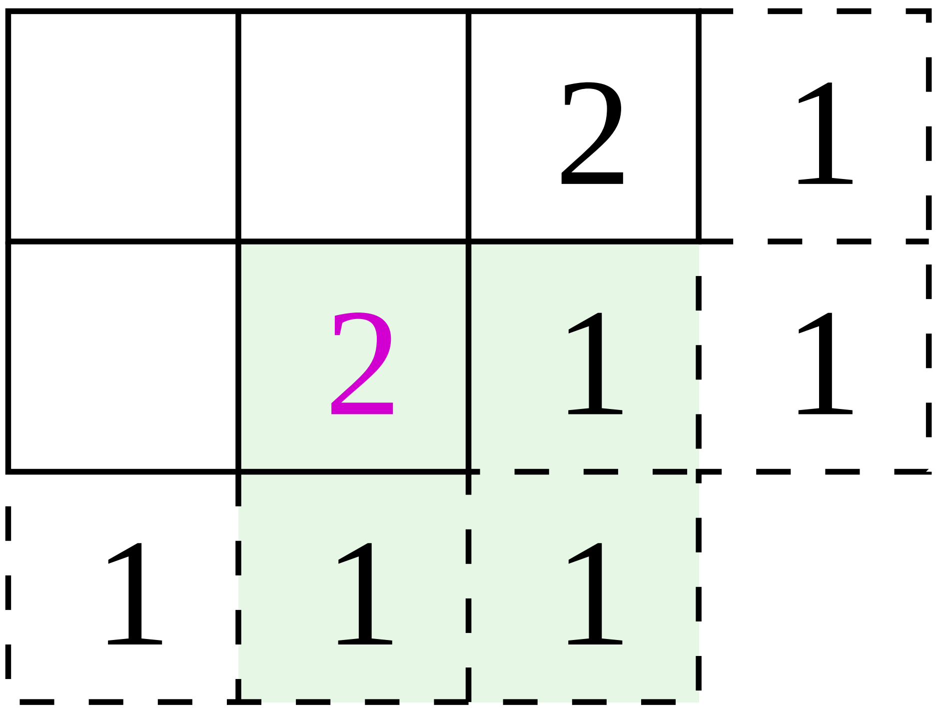

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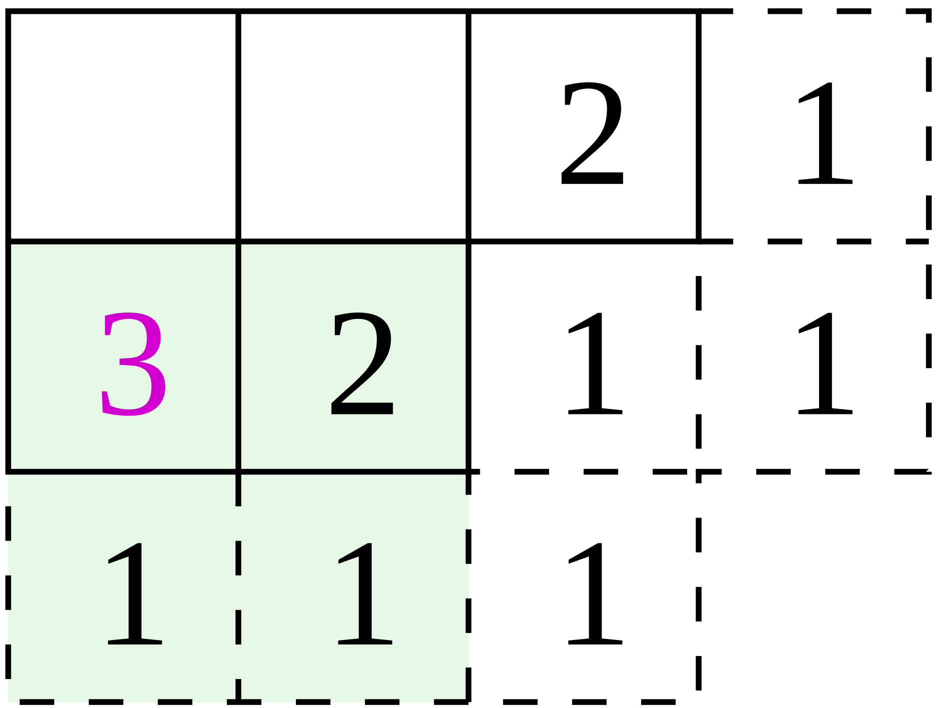

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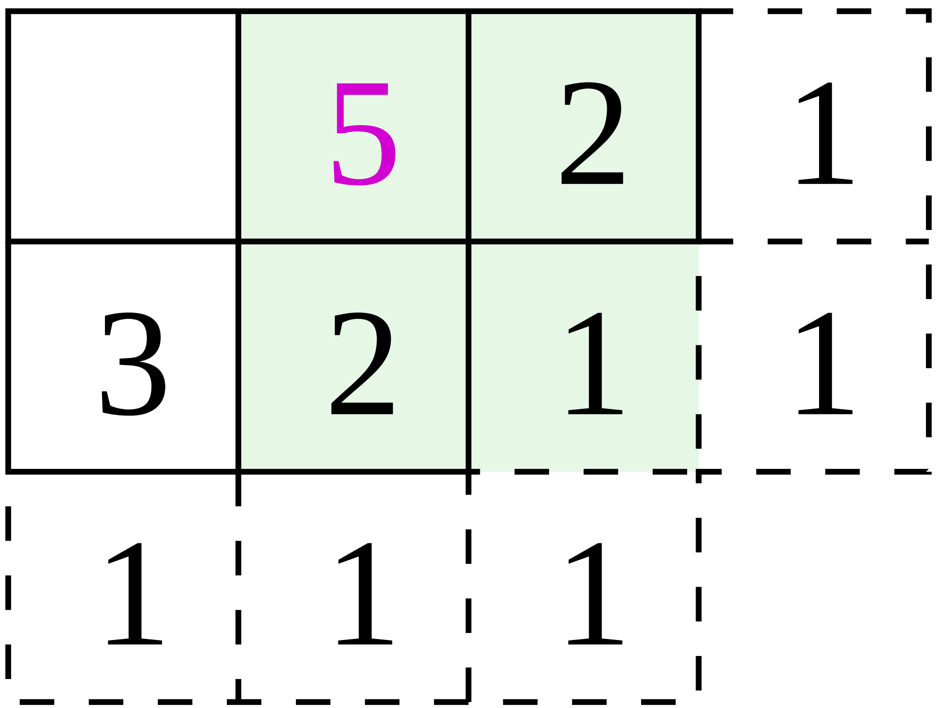

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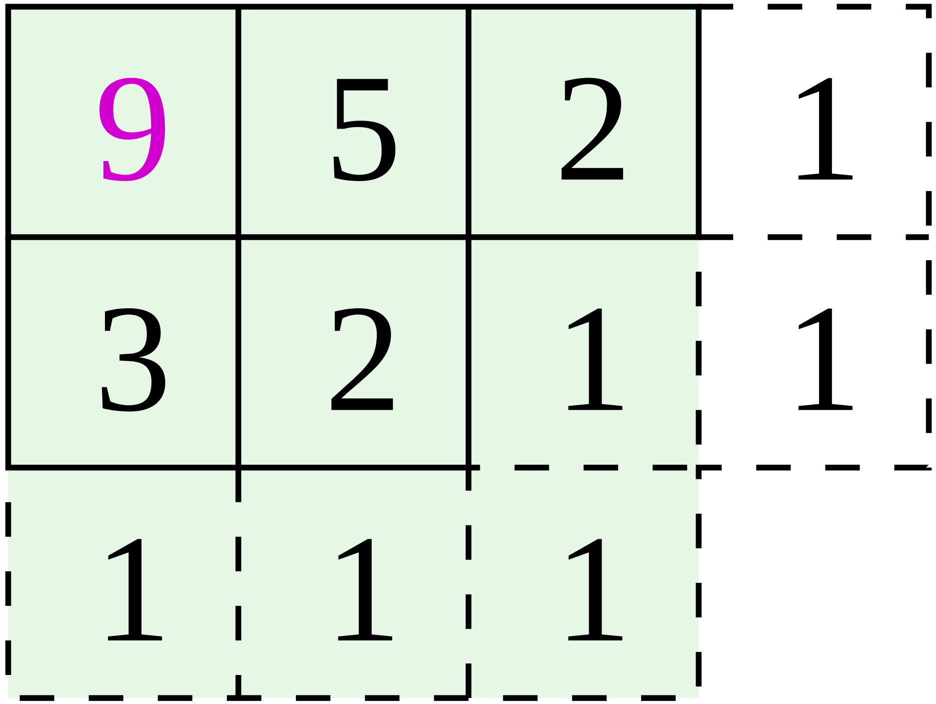

Uniqueness

Easy to see: the numbers n_t are well-defined and unique.

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Why? Expand $\det M_t$ by the first row. The coefficient of n_t is 1 by induction.

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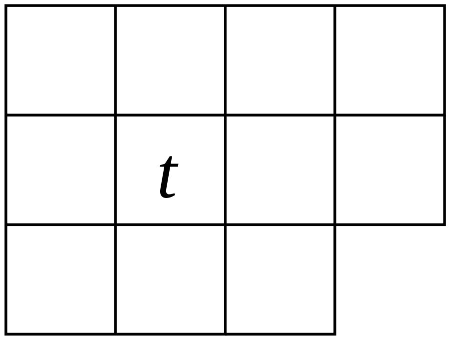

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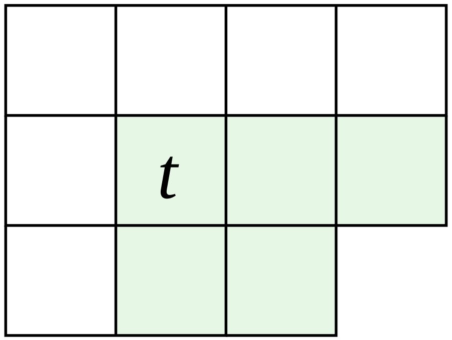

$\lambda = (4,4,3)$

$\lambda(t) = (3,2)$

$\boldsymbol{u_\lambda}$

$$u_\lambda = \#\{\mu : \mu \subseteq \lambda\}$$

u_λ

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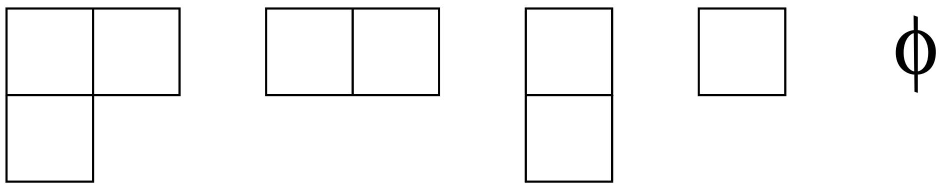
Example. $u_{(2,1)} = 5$:



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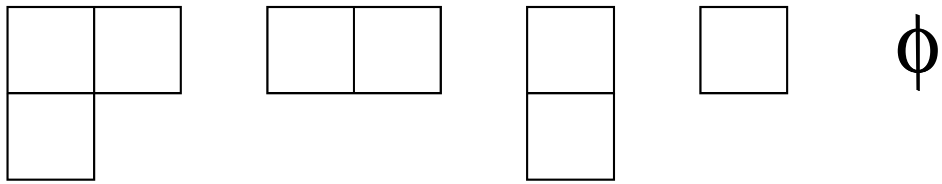

There is a determinantal formula for u_λ, due essentially to **MacMahon** and later **Kreweras** (not needed here).

Carlitz-Scoville-Roselle theorem

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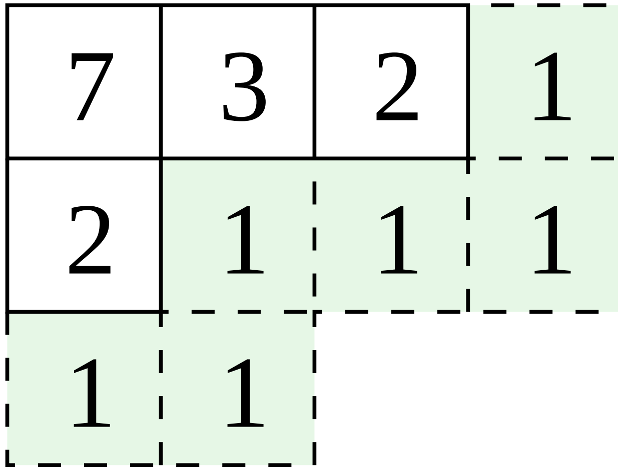
Theorem. $n_t = f(\lambda(t))$.

Proofs. 1. Induction (row and column operations).

2. Nonintersecting lattice paths.

An example



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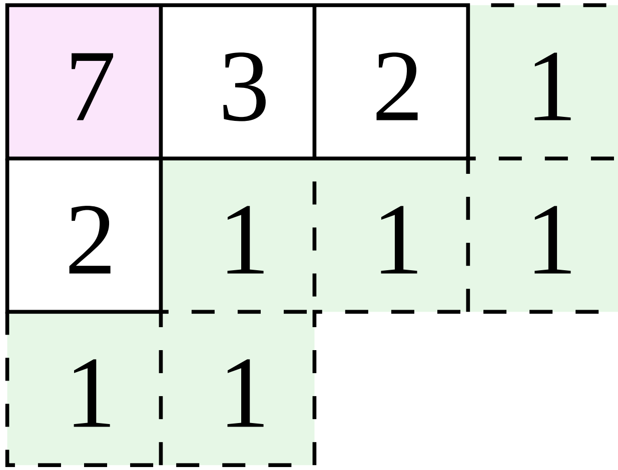

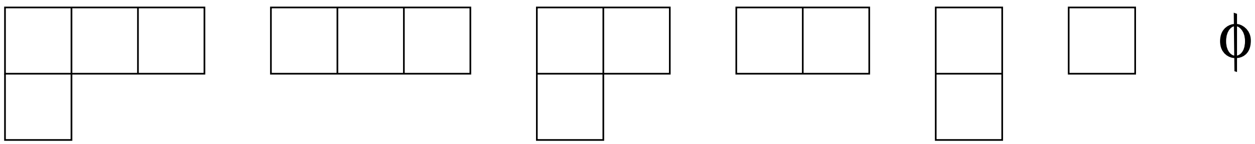

Smith normal form

$\boldsymbol{A}$: $n \times n$ matrix over commutative ring $\boldsymbol{R}$ (with 1)

Suppose there exist $\boldsymbol{P}, \boldsymbol{Q} \in \mathrm{GL}(n, R)$ such that

$$PAQ = B = \mathrm{diag}(d_1 d_2 \cdots d_n, d_1 d_2 \cdots d_{n-1}, \ldots, d_1),$$

where $d_i \in R$. We then call B a **Smith normal form (SNF)** of A.

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NOTE.

$$\mathrm{unit} \cdot \det(A) = \det(B) = d_1^n d_2^{n-1} \cdots d_n.$$

Thus SNF is a refinement of $\det(A)$.

Existence of SNF

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If R is a PID, such as $\mathbb{Z}$ or $K[x]$ ($K =$ field), then A has a unique SNF up to units.

Otherwise A "typically" does not have a SNF but may have one in special cases.

Algebraic interpretation of SNF

R: a PID

A: an $n \times n$ matrix over R with $\det(A) \neq 0$ and rows $v_1, \ldots, v_n \in R^n$

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Theorem.

$$R^n/(v_1, \ldots, v_n) \cong (R/e_1R) \oplus \cdots \oplus (R/e_nR).$$

An explicit formula for SNF

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Theorem. $e_{n-i+1}e_{n-i+2}\cdots e_n$ *is the gcd of all* $i \times i$ *minors of* A.

minor: determinant of a square submatrix.

Special case: e_n is the gcd of all entries of A.

Many indeterminates

For each square $(i, j) \in \lambda$, associate an indeterminate $\boldsymbol{x_{ij}}$ (matrix coordinates).

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x_{11}	x_{12}	x_{13}
x_{21}	x_{22}	

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$$u_\lambda(x) = \sum_{\mu \subseteq \lambda} \prod_{(i,j) \in \lambda/\mu} x_{ij}$$

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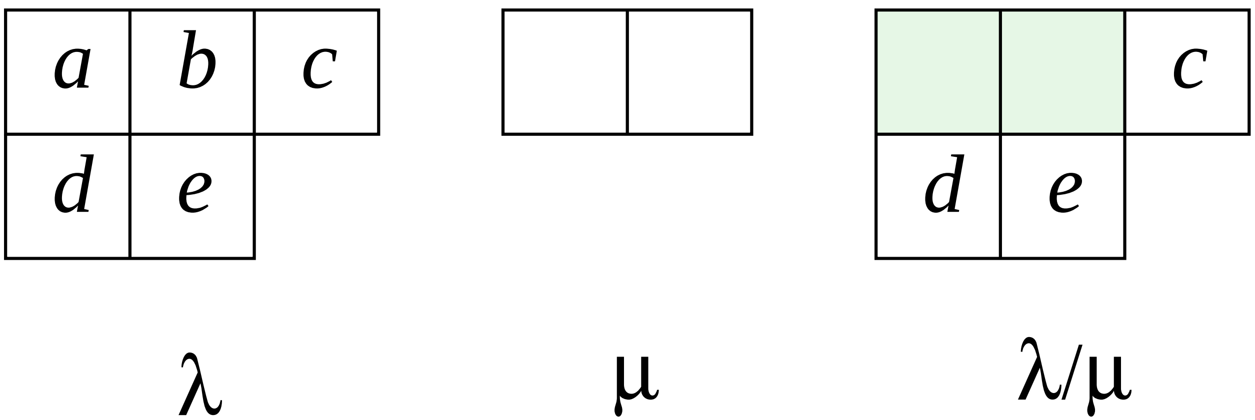

$$\prod_{(i,j) \in \lambda/\mu} x_{ij} = cde$$

An example

a	b	c
d	e	

$abcde+bcde+bce+cde+ce+de+c+e+1$	$bce+ce+c+e+1$	$c+1$	1
$de+e+1$	$e+1$	1	1
1	1	1	

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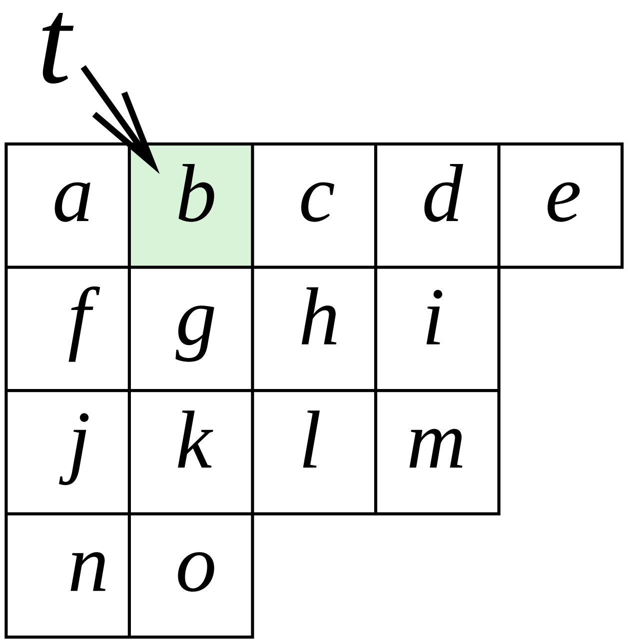

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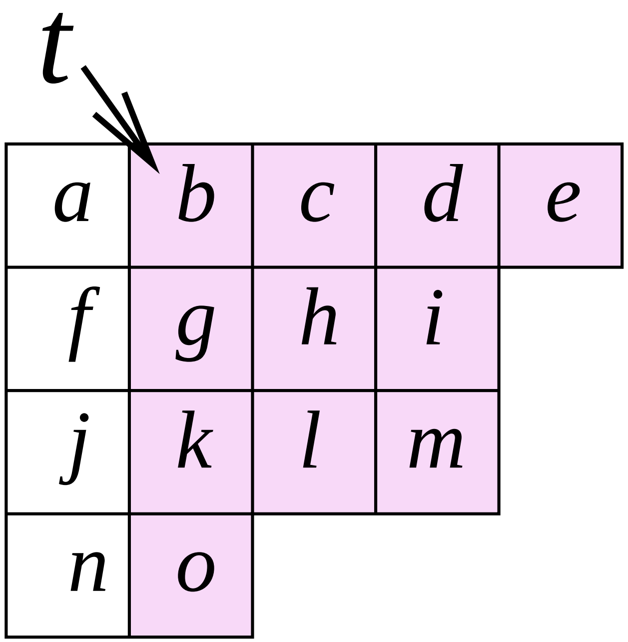

$$A_t = bcdeghiklmo$$

The main theorem

Theorem. *Let* $t = (i, j)$*. Then* M_t *has SNF*

$$\mathrm{diag}(A_{ij}, A_{i-1,j-1}, \ldots, 1).$$

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Proof. 1. Explicit row and column operations putting M_t into SNF.

2. (**C. Bessenrodt**) Induction.

An example

a	b	c
d	e	

$abcde+bcde+bce+cde$ $+ce+de+c+e+1$	$bce+ce+c$ $+e+1$	$c+1$	1
$de+e+1$	$e+1$	1	1
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$$\mathbf{SNF} = \mathrm{diag}(abcde, e, 1)$$

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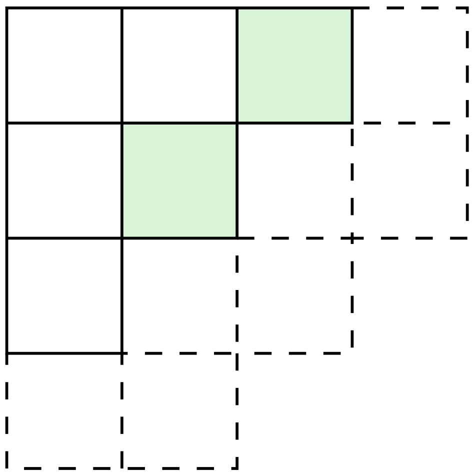

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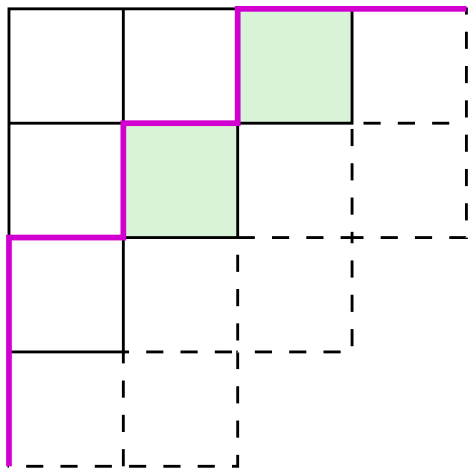
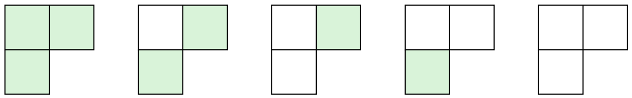


$u_{\delta_{n-1}}(x)\Big|_{x_{ij}=q}$ counts Dyck paths of length $2n$ by (scaled) area, and is thus the well-known q-analogue $\boldsymbol{C_n(q)}$ of the Catalan number C_n.

A q-Catalan example



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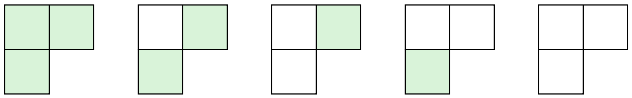

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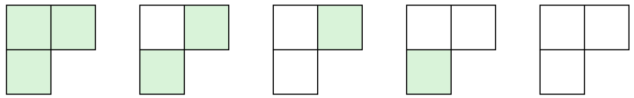

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- q-Catalan determinant previously known
- SNF is new

The second tidbit

A distributive lattice associated with three-term arithmetic progressions

Numberplay blog problem

New York Times Numberplay blog (March 25, 2013): Let $\boldsymbol{S} \subset \mathbb{Z}$, $\#S = 8$. Can you two-color S such that there is no monochromatic three-term arithmetic progression?

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Finally proved by **Noam Elkies**.

Compatible pairs

Elkies' proof is related to the following question:

Let $1 \leq i < j < k \leq n$ and $1 \leq a < b < c \leq n$.

$\{i, j, k\}$ and $\{a, b, c\}$ are **compatible** if there exist integers $x_1 < x_2 < \cdots < x_n$ such that x_i, x_j, x_k is an arithmetic progression and x_a, x_b, x_c is an arithmetic progression.

An example

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123 and 134 *are* compatible, e.g.,

$$(x_1, x_2, x_3, x_4) = (1, 2, 3, 5).$$

Elkies' question

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Example. When $n = 4$ there are eight such subsets $\mathcal{S}$:

$$\emptyset, \{123\}, \{124\}, \{134\}, \{234\},$$

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Not $\{123, 124\}$, for instance.

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Let $\boldsymbol{M_n}$ be the collection of all such $\mathcal{S} \subseteq \binom{[n]}{3}$, so for instance $\#M_4 = 8$.

Another example

Example. For $n = 5$ one example is

$$\mathcal{S} = \{123, 234, 345, 135\} \in M_5,$$

achieved by $1 < 2 < 3 < 4 < 5$.

Conjecture of Elkies

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A poset on M_n

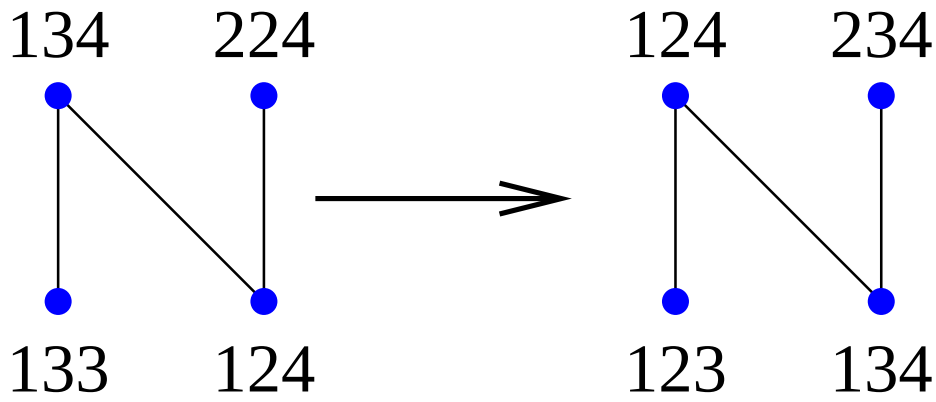
Jim Propp: Let Q_n be the subposet of $[n] \times [n] \times [n]$ (ordered componentwise) defined by

$$\boldsymbol{Q_n} = \{(i, j, k) \,:\, i + j < n + 1 < j + k\}.$$

antichain: a subset A of a poset such that if $x, y \in A$ and $x \leq y$, then $x = y$

There is a simple bijection from the antichains of Q_n to M_n induced by $(i, j, k) \mapsto (i, n + 1 - j, k)$.

The case $n = 4$



$(i, j, k) \longrightarrow (i, 5-j, k)$

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Order ideals

order ideal: a subset I of a poset such that if $y \in I$ and $x \leq y$, then $x \in I$

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$\boldsymbol{J(P)}$: set of order ideals of P, ordered by inclusion (a distributive lattice)

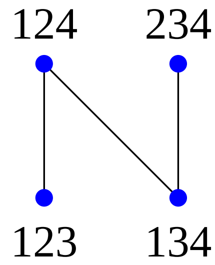
Join-irreducibles

join-irreducible of a finite lattice L: an element y such that exactly one element x is maximal with respect to $x < y$ (i.e., y **covers** x)

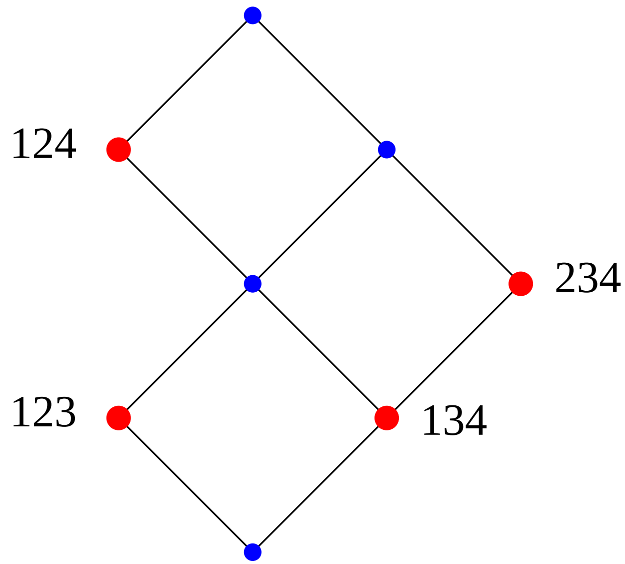
Theorem (FTFDL). *If L is a finite distributive lattice with the subposet P of join-irreducibles, then $L \cong J(P)$.*

The case $n = 4$

$P = Q_4$



$J(P) = M_4$



A partial order on M_n

Recall: there is a simple bijection from the antichains of Q_n to M_n induced by $(i, j, k) \mapsto (i, n+1-j, k)$.

Also a simple bijection from antichains of a finite poset to order ideals.

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Hence we get a bijection $J(Q_n) \to M_n$ that induces a distributive lattice structure on M_n.

Semistandard tableaux

T: semistandard Young tableau of shape of shape $\delta_{n-1} = (n-2, n-3, \ldots, 1)$, maximum part $\leq n-1$

1	1	2	5
2	3	3	
4	4		
5			

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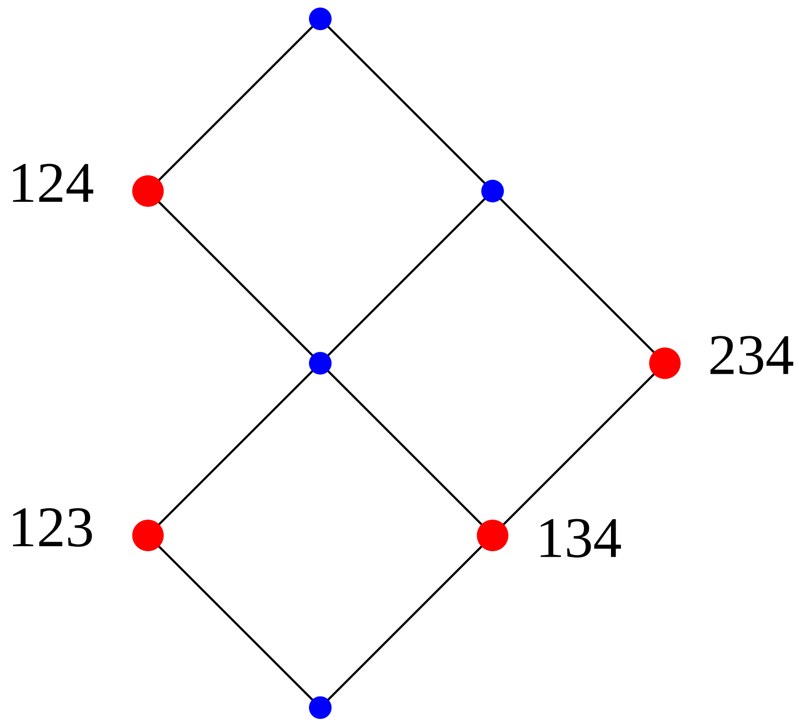
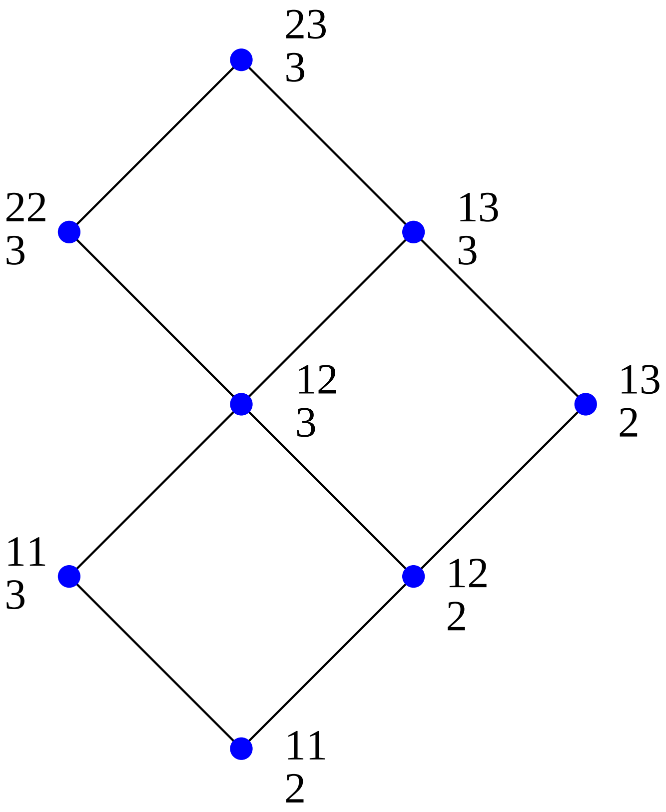
L_n: poset of all such T, ordered componentwise (a distributive lattice)

L_4 and M_4 compared



M_4



L_4

$L_n \cong M_n$

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Proof. Show that the poset of join-irreducibles of L_n is isomorphic to Q_n. $\square$

$\#L_n$

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Proof. $\#L_n = s_{\delta_{n-2}}(\underbrace{1,1,\ldots,1}_{n-1})$. Now use hook-content formula. $\square$

In fact,

$$s_{\delta_{n-2}}(x_1,\ldots,x_{n-1}) = \prod_{1\leq i<j\leq n-1}(x_i+x_j).$$

Maximum size elements of M_n

$f(n)$: size of largest element $\mathcal{S}$ of M_n.

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$$M_4 = \{\emptyset, \{123\}, \{124\}, \{134\}, \{234\},$$

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Thus $f(4) = 2$.

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Thus $f(4) = 2$.

Since elements of M_n are the antichains of Q_n, $f(n)$ is also the number of maximum size antichains of Q_n.

Evaluation of $f(n)$

Easy result (Elkies):

$$f(n) = \begin{cases} m^2, & n = 2m+1 \\ m(m-1), & n = 2m. \end{cases}$$

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Conjecture #2 (**Elkies**). Let $\boldsymbol{g(n)}$ be the number of antichains of Q_n of size $f(n)$. (E.g., $g(4) = 3$.) Then

$$g(n) = \begin{cases} 2^{m(m-1)}, & n = 2m+1 \\ 2^{(m-1)(m-2)}(2^m - 1), & n = 2m. \end{cases}$$

Maximum size antichains

$\boldsymbol{P}$: finite poset with largest antichain of size $\boldsymbol{m}$

$\boldsymbol{J(P)}$: lattice of order ideals of P

$\boldsymbol{D(P)} := \{x \in J(P)\ :\ x \text{ covers } m \text{ elements}\}$ (in bijection with m-element antichains of P)

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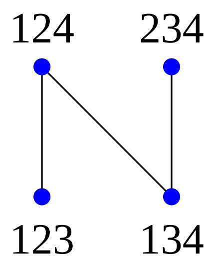
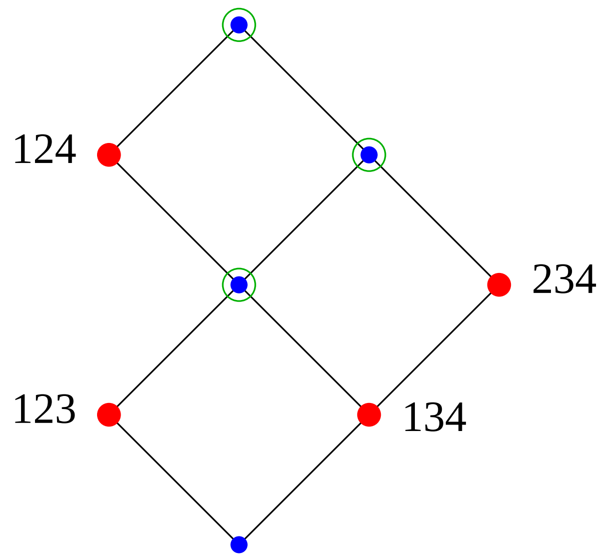
Easy theorem (**Dilworth**, 1960). $D(P)$ is a sublattice of $J(P)$ (and hence is a distributive lattice)

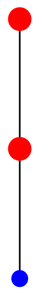
Example: M_4



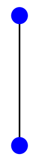
Q_4



$M_4 = J(Q_4)$



$D(M_4) = J(R_4)$



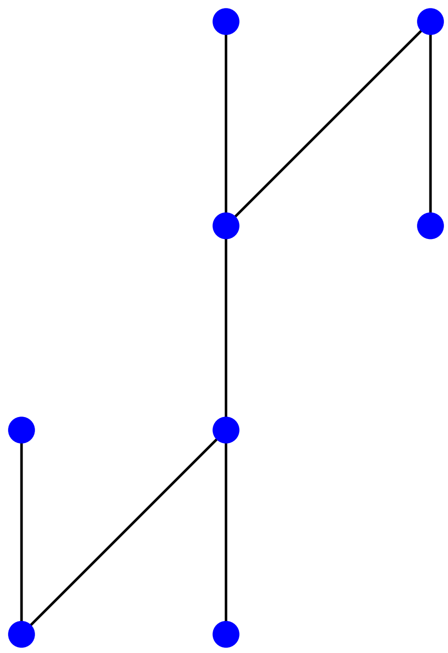
R_4

Application to Conjecture 2

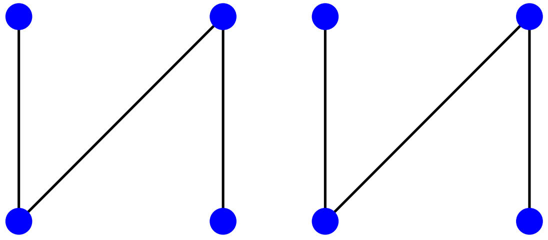
Recall: $g(n)$ is the number of antichains of Q_n of maximum size $f(n)$.

Hence $g(n) = \#D(Q_n)$. The lattice $D(Q_n)$ is difficult to work with directly, but since it is distributive it is determined by its join-irreducibles $\boldsymbol{R_n}$.

Examples of R_n



R_6



$R_7 \cong Q_4 + Q_4$

Structure of R_n

$\boldsymbol{n = 2m+1}$: $R_n \cong Q_{m+1} + Q_{m+1}$. Hence

$$g(n) = \#J(R_n) = \left(2^{\binom{m}{2}}\right)^2 = 2^{m(m-1)},$$

proving the Conjecture 2 of Elkies for n odd.

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Thus Conjecture 2 is true for all n.

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