Zeta functions of adjacency algebras induced by graphs

Mitsugu Hirasaka
Department of Mathematics
Pusan National University
Joint Work with Akihide Hanaki

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Γ: a finite digraph with v vertices
A_Γ: the adjacency matrix of Γ
R_Γ: the subring of $M_{v\times v}(\mathbb{Z})$ generated by A_Γ, i.e.,
$R_\Gamma = \{f(A_\Gamma) \mid f(x) \in \mathbb{Z}[x]\}$
L: an R_Γ-module, e.g., R_Γ, $\mathbb{Z}^v$

Questions
(1) How many submodules of L with given index are there?
(2) Find the class number $h(L)$, i.e., the number of isomorphism classes of submodules of L with finite index.

Example 1
Γ: the null graph
$R_\Gamma = \mathbb{Z}$
$L = \mathbb{Z}^m$

How many $\mathbb{Z}$-submodules of $\mathbb{Z}^2$ with index two are there?

You will see that $\{(2a, b) \mid a, b \in \mathbb{Z}\}$ and $\{(a, 2b) \mid a, b \in \mathbb{Z}\}$ are submodules of index two.

What else?

The number of $\mathbb{Z}$-submodules of $\mathbb{Z}^m$ with index n is equal to the number of integral lower triangular matrices $(b_{ij})_{1\leq i,j\leq m}$ such that
$0 \leq b_{ij} < b_{jj}$ for all i, j and $b_{11}b_{22}\cdots b_{mm} = n$.
For example, if $m = n = 2$, then the matrices are

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}.$$

Therefore,
$\mathbb{Z}(2,0) + \mathbb{Z}(0,1)$,
$\mathbb{Z}(1,0) + \mathbb{Z}(0,2)$ and
$\mathbb{Z}(1,1) + \mathbb{Z}(0,2)$ are the submodules of index two.

Class number

Since every submodule of $\mathbb{Z}^m$ with finite index is a free $\mathbb{Z}$-module of rank m, $h(\mathbb{Z}^m) = 1$.

Example 2

Γ: the path of length one

$$A_\Gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ R_\Gamma = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}$$

$L = \mathbb{Z}^2$

L has exactly one submodule of index 2, which is $M = \{(x, y) \in \mathbb{Z}^2 \mid x \equiv y \pmod 2\}$.

L has exactly two submodules of index 3, and

L has exactly three submodules of index 4.

Class number

Since M has exactly three submodules of index two, we have $M \not\simeq L$, and we can prove that $h(L) = 2$.

Example 3

Γ: the directed cycle of length 3

$$A_\Gamma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},\ R_\Gamma = \left\{ \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$$

$L = R_\Gamma$

L has exactly two submodules of index 2,

L has exactly one submodule of index 3, and

L has exactly three submodules of index 4.

Works by Louis Solomon

L. Solomon,
Zeta functions and integral representation theory, *Advances in Math.* 26 (1977), no. 3, 306–326.

A: a f.d. semisimple $\mathbb{Q}$-algebra
R: a $\mathbb{Z}$-order of A, (a subring of A, a $\mathbb{Z}$-lattice)
V: a f.g. A-module
L: an R-lattice in V (an R–module, $\mathbb{Z}$-lattice)
For $n \in \mathbb{Z}^+$ we denote by a_n the number of submodules of L with index n.

$$\zeta_L(s) := \sum_{n \geq 1} \frac{a_n}{n^s} = \prod_{p \in B} \delta_p(p^{-s}) \cdot \zeta_V(s)$$

for some $\delta_p(t) \in \mathbb{Q}(t)$ where B is a finite set of primes and
$\zeta_V(s) = \prod_{k=1}^{r} \prod_{j=0}^{m_k e_k - 1} \zeta_{F_k}(n_k s - j)$
when $A \simeq \bigoplus_{k=1}^{r} A_k$, $A_k \simeq M_{l_k}(D_k)$
$F_k = Z(A_k)$, $n_k^2 = \dim_{F_k} A_k$, $e_k = \dim_{F_k} D_k$
$V \simeq \bigoplus_{k=1}^{r} m_k W_k$,
$\zeta_{F_k}(s)$ is the Dedekind zeta function of F_k.

Example 1

$A = \mathbb{Q}$

$R = \mathbb{Z}$

$V = \mathbb{Q}^m$

$L = \mathbb{Z}^m$

$$\zeta_L(s) = \zeta(s)\zeta(s-1)\cdots\zeta(s-m+1)$$

where $\zeta(s)$ is the Riemann zeta function, i.e., $\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$.

If $m = 2$, then $a_2 = 3$, $a_3 = 4$ and $a_4 = 7$ since

$$\zeta(s)\zeta(s-1) = (\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\cdots)(\frac{1}{1^s}+\frac{2}{2^s}+\frac{3}{3^s}+\cdots)$$

$$= \frac{1}{1^s}+(2+1)\frac{1}{2^s}+(3+1)\frac{1}{3^s}+(4+2+1)\frac{1}{4^s}+\cdots.$$

Example 2
Γ: the path of length 2, $R = R_\Gamma$, $L = \mathbb{Z}^2$

$$\zeta_L(s) = (1 - 2^{-s} + 2^{1-2s})\zeta(s)\zeta(s)$$

$$= (1 - \frac{1}{2^s} + \frac{2}{4^s})(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots)(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots)$$

How many submodules with index 2?

Known Results
G: a finite group
$A = \mathbb{Q}[G]$: the group ring of G over $\mathbb{Q}$
$R = \mathbb{Z}[G]$: a $\mathbb{Z}$-order of $\mathbb{Q}[G]$
$V = A$, $L = R$

[L.Solomon]
If $G \simeq C_p$, then
$\zeta_L(s) = (1 - p^{-s} + p^{1-2s})\zeta(s)\zeta_F(s)$
where $F = \mathbb{Q}(e^{\frac{2\pi i}{p}})$ and p is a prime.

[I.Reiner]
$G \simeq C_{p^2}$

[Y.Hironaka]
G is meta-cyclic with certain conditions

[Y.Takegahara]
$G \simeq C_2 \times C_2$ or $C_3 \times C_3$.

Attempts for Several Digraphs

In this slide we consider zeta functions of $L = R_\Gamma$ for several digraphs Γ.

(1) Γ: the direct cycle of length N
$R_\Gamma \simeq \mathbb{Z}[C_N]$ where $C_N \simeq \langle(12\cdots N)\rangle$.

(2) Γ: the cycle of length N
It is difficult to compute the zeta function of R_Γ even if $N = 4$. But, it is possible whenever N is a prime. [H, Hanaki]

(3) $\Gamma = Cay(\mathbb{F}_q, H)$ where $H \leq F_q^\times$
It is difficult to compute the zeta function of R_Γ even if $q = 4$. But, it is possible whenever q is a prime. [H, Hanaki]

(4) Γ: the complete graph of degree N
We can find it for each N. [H, Hanaki]

Theorem 1
Let (X, S) be an association scheme such that $|X|$ is a prime p. Then

$$\zeta_{\mathbb{Z}S}(s) = (1 - p^{-s} + p^{1-2s})\zeta(s)\zeta_F(s)$$

where F is the minimal splitting field of a non-principal character of $\mathbb{C}S$.

Theorem 2
Let (X, S) be an association scheme such that $|S| = 2$ and $|X| = \prod_{i=1}^{k} {p_i}^{m_i}$. Then

$$\zeta_{\mathbb{Z}S}(s) = \prod_{i=1}^{k} \delta_{p_i,m_i}({p_i}^{-s}) \cdot \zeta(s)^2$$

where $\delta_{p_i,m_i}(t) = {p_i}^{m_i} t^{2m_i} + \sum_{j=0}^{m_i-1} {p_i}^{j} t^{2j}(1-t)$.

Remarks
(1) If $\Gamma = Cay(\mathbb{F}_p, H)$ with $H \leq \mathbb{F}_p^{\times}$, then $R_\Gamma = \mathbb{Z}S$.
(2) If $\Gamma = K_{|X|}$, then $R_\Gamma = \mathbb{Z}S$ with $|S| = 2$.

What is an association scheme?

A set S of $N \times N$ $\{0,1\}$-matrices $\{A_0, A_1, \ldots, A_d\}$ such that
(i) $\sum_{i=0}^{d} A_i = J$ where J is the all one matrix;
(ii) A_0 is the identity matrix;
(iii) $\{A_0^T, A_1^T, \ldots, A_d^T\} = \{A_0, A_1, \ldots, A_d\}$;
(iv) $\forall i, j,\ A_i A_j \in \mathrm{span}_{\mathbb{Z}}\{A_0, A_1, \ldots, A_d\}$.

Examples

(i) $\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\}$

(ii) $\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \right\}$

(iii) The centralizer ring of a transitive permutation group has a unique basis consisting of $\{0,1\}$-matrices, which satisfies the above conditions.

Definition (Adjacency Algebras)
$\mathrm{span}_{\mathbb{Z}}\{A_0, A_1, \ldots, A_d\}$ is a $\mathbb{Z}$-algebra, denoted by $\mathbb{Z}S$.

For a ring R we denote $R \bigotimes \mathbb{Z}S$ by RS, called the *adjacency algebra* of S over R.

Sketch of the proof of Theorem 1

Let (X,S) be an association scheme such that $|X|$ is a prime p.

Recall that

$$\zeta_L(s) = \prod_{q\in B} \delta_q(q^{-s}) \cdot \zeta_V(s).$$

It is known that $\mathbb{Q}S \simeq \mathbb{Q} \bigoplus F$
where F is the minimal splitting field of $\mathbb{C}S$.
Thus, $\zeta_V(s) = \zeta(s) \cdot \zeta_F(s)$.

Lemma 1 We have $B = \{p\}$.
(Proof) Let $y \in \Lambda$.
Then $y = \sum_{i=0}^{d} c_i A_i$ for $c_0, c_1, \ldots, c_d \in \mathbb{Q}$.
Note that $c_i = \rho(y)/(n_i|X|)$ where
n_i is the constant rowsum of A_i and
ρ is the standard character.
Since $\chi_0 = \sum_{i=0}^{d} m_i \chi_i$, $m_0 = n_0 = 1$ and $n_1 = n_2 = \cdots = n_d = m_1 = m_2 = \cdots = m_d$, we have $py \in \mathbb{Z}S$. This implies that p is a unique prime which divides $|\Lambda : \mathbb{Z}S|$.

Lemma 2 [Solomon]

$$\delta_p(p^{-s}) = \frac{\zeta_{\mathbb{Z}_p S}(s; \mathbb{Z}_p S)}{\zeta_{\mathbb{Q}S}(s)_p}$$

where $\mathbb{Z}_p$ is the localization of $\mathbb{Z}$ at p.

Lemma 3
$\zeta_{\mathbb{Q}S}(s)_p = \zeta(s)_p \zeta_F(s)_p = (1 + p^{-s} + p^{-2s} + \cdots)^2$.
(Proof) Let R be the ring of integers of F. It is known ([Hanaki]) that there exists a unique maximal ideal I which divides pR, and $|R : I| = p$. Thus, the assertion follows from the definition of the Dedekind zeta function.

Lemma 4 We have the following:
(1) $\mathbb{Z}pS$ has exactly one maximal submodule M;
(2) M has exactly $(p+1)$ maximal submodules, two of which are isomorphic to M and
$p-1$ of which are isomorphic to $\mathbb{Z}_pS$.
(Proof)
(1) follows from the fact that $(\mathbb{Z}_p/p\mathbb{Z}_p)S$ is local.
(2) First, we prove that
M has a $\mathbb{Z}_p$-basis $(u, v, v^2, \ldots, v^d)$ where
$u = \sum_{i=0}^{d} A_i$ and $v = kA_0 - A_i$ for a suitable A_i,
Then we can prove that $\{N_a \mid a = 0, 1, \ldots, p, \infty\}$ are exactly maximal submodules of M where N_a is spanned by $(pu, au+v, v^2, \ldots, v^d)$ for $a \neq \infty$ and N_∞ is spanned by $(u, pv, v^2, \ldots, v^d)$.
Moreover, $N_0 \simeq N_\infty \simeq M$.

Lemma 5
We have $\delta_p(X) = 1 - X + pX^2$.
(Proof) The structure of the poset of submodules of $\mathbb{Z}_pS$ is the same as $\mathbb{Z}_p[G]$ where G is a group of order p. So, the assertion follows from the method given in [Solomon].

Other digraphs

(1) Γ: The directed path of length N
Then $\mathbb{Q} \otimes R_\Gamma$ is not semisimple and $h(R_\Gamma)$ is not finite.

(2) Γ: the cycle of length 4
It is expected that $h(R_\Gamma) = 10$
(joint work with Semin Oh).

(3) Γ: a strongly-regular graph with integral eigenvalues, e.g., the Peterson graph.
$\mathbb{Q} \otimes R_\Gamma \simeq \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}$

(4) Γ: the path of length N

(5) Γ: the star graph with N leaves.

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Thank you for your attention.