Jacobi-Stirling numbers and Jacobi-Stirling permutations

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• Y. Gelineau, J. Z., Combinatorial interpretations for the Jacobi-Stirling numbers, EJC, 2010.

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- I. Gessel, Z.C. Lin, J. Z., Jacobi-Stirling polynomials and P-partitions, European JC, 2012.

Jacobi-Stirling numbers

Introduction

Jacobi polynomials $P_n^{(\alpha,\beta)}(t)$ satisfy the differential equation :

$$(1-t^2)y''(t)+(\beta-\alpha-(\alpha+\beta+2)t)y'(t)+n(n+\alpha+\beta+1)y(t)=0.$$

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Let $\ell_{\alpha,\beta}[y](t)$ be the differential operator of Jacobi :

$$\ell_{\alpha,\beta}[y](t)=\frac{1}{(1-t)^{\alpha}(1+t)^{\beta}}\left(-(1-t)^{\alpha+1}(1+t)^{\beta+1}y'(t)\right)'$$

Then $P_n^{(\alpha,\beta)}(t)$ is an eigenfunction of $\ell_{\alpha,\beta}[y](t)$:

$$\ell_{\alpha,\beta}[y](t)=n(n+\alpha+\beta+1)y(t)$$

Everitt-Kwon-Littlejohn-Wellman-Yoon (2007) proved the expansion of the *n*-th composition of $\ell_{\alpha,\beta}$:

$$(1-t)^{\alpha}(1+t)^{\beta}\ell^{n}_{\alpha,\beta}[y](t)=\sum_{k=0}^{n}(-1)^{k}\left(\mathrm{JS}(n,k;\alpha+\beta+1)(1-t)^{\alpha+k}(1+t)^{\beta+k}y^{(k)}(t)\right)^{(k)},$$

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where $\mathrm{JS}(n,k;z)$ are called the Jacobi-Stirling numbers of the second kind defined by

$$X^{n}=\sum_{k=0}^{n}\mathrm{JS}(n,k;z)\prod_{i=0}^{k-1}(X-i(z+i)).$$

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Equivalently these numbers can be defined by the recurrence :

$$\mathrm{JS}(n,k;z) = \mathrm{JS}(n-1,k-1;z) + k(k+z)\,\mathrm{JS}(n-1,k;z), \quad n,k \geq 1,$$

$$\mathrm{JS}(0,0;z) = 1, \qquad \mathrm{JS}(n,k;z) = 0 \quad \text{if } k \notin \{1,\ldots,n\}.$$

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When $z=1$, $\mathrm{LS}(n,k):=\mathrm{JS}(n,k;1)$ are called Legendre-Stirling numbers.

Jacobi-Stirling numbers

$k \backslash n$	0	1	2	3	4
0	1	0	0	0	0
1		1	$1+z$	$1+2z+z^2$	$1+3z+3z^2+z^3$
2			1	$5+3z$	$21+24z+7z^2$
3				1	$14+6z$
4					1

Recall : Stirling numbers of second kind

The **Stirling numbers (of second kind)** $S(n,k)$ are defined by the relation :

$$S(n,k) = S(n-1,k-1) + kS(n-1,k).$$

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They count :
partitions of $[n] := \{1, 2, \ldots, n\}$ into k blocks,
For example,

$$\pi = \Big\{\{1,3,6\},\{2,5\},\{4\}\Big\}$$

is a partition of [6] in 3 blocks.

Central factorial numbers

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They count (Dumont, 1974) :

- the pairs (π_1, π_2) of partitions of $[n]$ into k blocks, with $\min(\pi_1) = \min(\pi_2)$,
- the number of partitions of $\{1, 1', 2, 2', \ldots, n, n'\}$ in k blocks such that, for each block B, if i is the smallest integer with $i \in B$ or $i' \in B$, then both i and i' are in B.

Example

For example, for the two partitions of [6] in 3 blocks,

$$\pi_1 = \big\{\{1,6\},\{2,3,5\},\{4\}\big\}, \quad \pi_2 = \big\{\{1,5\},\{2,3\},\{4,6\}\big\},$$

we have $\min(\pi_1) = \min(\pi_2) = \{1,2,4\}$.
It corresponds to the partition π of $\{1,1',2,2',\dots,6,6'\}$ into 3 blocks,

$$\pi = \big\{\{1,1',5',6\},\{2,2',3,3',5\},\{4,4',6'\}\big\}$$

Jacobi-Stirling numbers :

$$\mathrm{JS}(n,k;z)=\mathrm{JS}(n-1,k-1;z)+k(k+z)\,\mathrm{JS}(n-1,k;z),\qquad n,k\geq 1.$$

Jacobi-Stirling numbers :

$$\mathrm{JS}(n,k;z) = \mathrm{JS}(n-1,k-1;z) + k(k+z)\,\mathrm{JS}(n-1,k;z), \qquad n,k \geq 1.$$

$\Longrightarrow$ $\mathrm{JS}(n,k;z)$ is a polynomial in z of degree $n-k$:

$$\mathrm{JS}(n,k;z) = a_{n,k}^{(0)} + a_{n,k}^{(1)} z + \cdots + a_{n,k}^{(n-k)} z^{n-k}.$$

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Moreover,

$$a_{n,k}^{(n-k)} = S(n,k),$$

$$a_{n,k}^{(0)} = U(n,k).$$

Legendre-Stirling numbers

Definition (Andrews-Littlejohn, 2009)

A signed k-**partition** *of* $[\pm n]_0 = \{0, \pm 1, \pm 2, \ldots, \pm n\}$ *is a partition of* $[\pm n]_0$ *into* $k+1$ *blocks* $B_0, B_1, \ldots, B_k$, *such that*

- $0 \in B_0$ *and* $\forall i \in [n], \{i, -i\} \not\subset B_0$,
- $\forall j \in [k]$, i *is the smallest integer* > 0 *such that* $i \in B_j$ *or* $-i \in B_j \Leftrightarrow \{i, -i\} \subset B_j$.

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For example,

$$\pi = \{\{0, 2, -5, -6\}, \{\pm 1, -2, 6\}, \{\pm 3\}, \{\pm 4, 5\}\}$$

is a signed 3-partition of $[\pm 6]_0$.

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Theorem (Andrews-Littlejohn, 2009)

*The Legendre-Stirling number $LS(n, k)$ is the number of signed k-**partitions** of $[\pm n]_0$.*

Jacobi-Stirling numbers

Theorem (Gelineau-Z., 2009)

The coefficient $a_{n,k}^{(i)}$ is equal to the number of signed k-partitions of $[\pm n]_0$ with i negative numbers in B_0.

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Proof : Since $\mathrm{JS}(n,k;z)$ satisfy the recurrence relation :

$$\mathrm{JS}(n,k;z) = \mathrm{JS}(n-1,k-1;z) + k(k+z)\,\mathrm{JS}(n-1,k;z),$$

it suffices to verify that the numbers satisfy the same recurrence:

$$a_{n,k}^{(i)} = a_{n-1,k-1}^{(i)} + k a_{n-1,k}^{(i-1)} + k^2 a_{n-1,k}^{(i)}.$$

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it suffices to verify that the numbers satisfy the same recurrence:

$$a_{n,k}^{(i)} = \underbrace{a_{n-1,k-1}^{(i)}}_{B_k=\{\pm n\}} + \underbrace{ka_{n-1,k}^{(i-1)}}_{-n\in B_0} + \underbrace{k^2 a_{n-1,k}^{(i)}}_{\text{otherwise}}.$$

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Jacobi-Stirling numbers

$a_{n,k}^{(i)} = \sharp\{\text{signed k- partitions } [\pm n]_0 \text{ with } i \text{ terms} < 0 \text{ in } B_0\}$

Jacobi-Stirling numbers

$a_{n,k}^{(i)} = \sharp\{$signed k- partitions $[\pm n]_0$ with i terms < 0 in $B_0\}$

$\Rightarrow$ For $i = n - k$, we recover the interpretation of $S(n,k)$.

$$\pi = \big\{\{0, -3, -5, -6\}, \{\pm 1, 3, 6\}, \{\pm 2, 5\}, \{\pm 4\}\big\}$$

$$\Downarrow$$

$$\pi' = \big\{\{1, 3, 6\}, \{2, 5\}, \{4\}\big\}$$

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$$\pi = \big\{\{0,-3,-5,-6\},\{\pm 1,3,6\},\{\pm 2,5\},\{\pm 4\}\big\}$$

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$$\pi' = \big\{\{1,3,6\},\{2,5\},\{4\}\big\}$$

$\Rightarrow$ For $i = 0$, we recover the interpretation of $U(n,k)$.

$$\pi = \big\{\{0,3,6\},\{\pm 1,-3\},\{\pm 2,-5\},\{\pm 4,5,-6\}\big\}$$

$$\Downarrow$$

$$\pi = \{\{\pm 1,\pm 3\},\{\pm 2,-5\},\{\pm 4,5,\pm 6\}\}.$$

The **Jacobi-Stirling numbers of first kind** $\mathrm{js}(n,k;z)$ are defined by reversing the connection relation for $\mathrm{JS}(n,k;z)$:

$$X^n = \sum_{k=0}^{n} \mathrm{JS}(n,k;z) \prod_{i=0}^{k-1} (X - i(z+i)),$$

$$\prod_{i=0}^{n-1} (X - i(z+i)) = \sum_{k=0}^{n} (-1)^{n-k} \mathrm{js}(n,k;z) X^k.$$

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These numbers satisfy then the recurrence relation :

$$\mathrm{js}(n, k; z) = \mathrm{js}(n - 1, k - 1; z) + (n - 1)(n - 1 + z)\,\mathrm{js}(n - 1, k; z).$$

$k \backslash n$	0	1	2	3	4
0	1	0	0	0	0
1		1	$z+1$	$2z^2+6z+4$	$6z^3+36z^2+66z+36$
2			1	$3z+5$	$11z^2+48z+49$
3				1	$6z+14$
4					1

The **Stirling numbers (of first kind)** $s(n, k)$:

$$s(n, k) = s(n-1, k-1) + (n-1)s(n-1, k).$$

The **central factorial numbers (of first kind)** $u(n, k)$:

$$u(n, k) = u(n-1, k-1) + (n-1)^2 u(n-1, k).$$

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$$u(n,k) = u(n-1,k-1) + (n-1)^2 u(n-1,k).$$

$s(n,k)$ counts the number of permutations of $[n]$ with k cycles.

Theorem

$js(n, k; z)$ is a polynomial in z with degree $n - k$:

$$js(n, k; z) = b_{n,k}^{(0)} + b_{n,k}^{(1)} z + \cdots + b_{n,k}^{(n-k)} z^{n-k}.$$

Moreover,

$$b_{n,k}^{(n-k)} = s(n, k),$$

$$b_{n,k}^{(0)} = u(n, k).$$

Combinatorial interpretations

Let $\Sigma(n, k) :=$ all (σ, τ) such that

- σ is a permutation of $\{0, 1, \ldots, n\}$, τ is a permutation of $\{1, \ldots, n\}$, and both have k cycles.
- 1 and 0 are in the same cycle in σ.
- Among their nonzero entries, σ and τ have the same cyclic minima.

Theorem (Gelineau-Z., 2009)

$js(n, k; z)$ is the enumerative polynomial in z for $\Sigma(n, k)$ with respect to the number of non-zero left-to-right minima in the cycle containing 0 in σ, written as a word beginning with $\sigma(0)\sigma^2(0)\ldots$.

Jacobi-Stirling posets and permutations

Diagonal generating functions

For fixed k, consider the Diagonal sequence of Jacobi-Stirling numbers $\{JS(n+k, n; z)\}_{n\geq 0}$:

$$\mathrm{JS}(n+k, n; z) = p_{k,0}(n) + p_{k,1}(n)z + \cdots + p_{k,k}(n)z^k$$

Theorem (G-L-Z)

There is a polynomial $A_{k,i}(t) \in \mathbb{N}[t]$ *of degree* $2k-i$, *with* $A_{k,i}(0) = 0$ *such that*

$$\sum_{n\geq 0} p_{k,i}(n)t^n = \frac{A_{k,i}(t)}{(1-t)^{3k-i+1}}$$

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Problem: What do the coefficients of $A_{k,i}$ mean?

First values

$k \backslash i$	0	1	2
0	$\dfrac{1}{1-t}$		
1	$\dfrac{t+t^2}{(1-t)^4}$	$\dfrac{t}{(1-t)^3}$	
2	$\dfrac{t+14t^2+21t^3+4t^4}{(1-t)^7}$	$\dfrac{2t+12t^2+6t^3}{(1-t)^6}$	$\dfrac{t+2t^2}{(1-t)^5}$

Stanley's P-partition theory

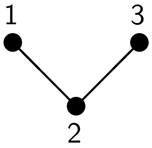
Let P be a poset on $[k]$ with partial order $\prec$. A P-partition is a function $f : [k] \to \mathbb{N}$ such that

(i) if $i \prec j$ then $f(i) \leq f(j)$

(ii) if $i \prec j$ and $i > j$ then $f(i) < f(j)$.

Example

Consider the following poset:

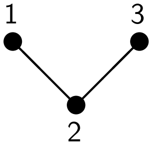


We have $2 \prec 3$ and $2 \prec 1$ and $2 > 1$. Hence a P-partition of the poset is a function f satisfies $f(2) \leq f(3)$ and $f(2) < f(1)$.

A linear extension of a poset P is an extension of P to a total order. Denote by $\mathscr{L}(P)$ the set of all linear extensions of P.

Example

For the following poset



we have $\mathscr{L}(P) = \{213, 231\}$.

The P-partitions of a poset P can be refined by the linear extensions of P.

Lemma

Let $\mathscr{A}(P)$ be the set of all P-partitions of a poset P. We have the disjoint union

$$\mathscr{A}(P) = \coprod_{\pi \in \mathscr{L}(P)} \mathscr{A}(\pi).$$

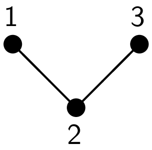
(Sketch of the proof) This can be proved by induction on the number of incomparable pairs in P.

Stanley's P-partition theory

The order polynomial $\Omega_P(n)$ is the number of P-partitions of a poset P with all parts in $[n]$.

Example

1. We have $\Omega_P(n) = 2\binom{n+1}{3}$ for the following poset.



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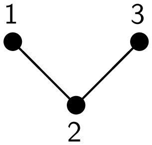

2. If P is the antichain on $[k]$ then $\Omega_P(n) = n^k$.

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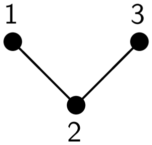


2. If P is the antichain on $[k]$ then $\Omega_P(n) = n^k$.
3. If P is the (natural ordered) chain on $[k]$ then $\Omega_P(n) =$ the number of k-multisets on [n] $= \binom{n+k-1}{k}$.

Theorem (P-partition theory)

Let P be a poset on $[k]$. Then we have

$$\sum_{n\geq 0}\Omega_P(n)t^n=\frac{\sum_{\pi\in\mathscr{L}(P)}t^{des(\pi)+1}}{(1-t)^{k+1}}.$$

(Sketch of proof) By the Lemma, it is enough to consider the case in which P is a chain on $[k]$ and reduce to the binomial theorem:

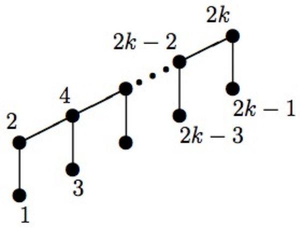
$$\sum_{n\geq 0}\binom{n+k-1}{k}t^n=\frac{t}{(1-t)^{k+1}}. \qquad \square$$

Example 1 (Antichain)
If P is the antichain on $[k]$, we have

$$\sum_{n\geq 0} n^k t^n = \frac{\sum_{i=1}^{k} A(k,i) t^i}{(1-t)^{k+1}},$$

where the Eulerian number $A(k,i)$ counts the number of permutations of $[k]$ with $i-1$ descents.

Stirling poset



Example 2 (Stirling poset, Park 1994)
Consider the above poset on $[2k]$, we have

$$\Omega_P(n) = \sum_{1 \le f(2) \le \cdots \le f(2k) \le n} f(2)f(4)\cdots f(2k)$$
$$= [t^k]\frac{1}{(1-t)(1-2t)\cdots(1-nt)}$$
$$= S(n+k, n).$$

Stirling permutations

Theorem (Park, 1994)

$$\sum_{n\geq 0} S(n+k,n)t^n = \frac{\sum_{j=1}^{k} c_{k,j} t^j}{(1-t)^{2k+1}},$$

where $c_{k,j}$ counts the linear extensions of Stirling poset with $j-1$ descents.

Stirling permutations

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where $c_{k,j}$ counts the linear extensions of Stirling poset with $j-1$ descents.

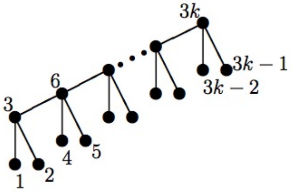
Stirling permutation: a permutation of $\{1,1,2,2,\dots,n,n\}$ such that, for each i, $1\leq i\leq n$, the elements occurring between two occurrences of i are at least i.

Example

$2211, 1221, 1122$ are all the Stirling permutations of $\{1,1,2,2\}$.

Theorem (Gessel and Stanley, 1978)

The integer $c_{k,j}$ counts the number of Stirling permutations of $\{1,1,2,2,\dots,k,k\}$ with $j-1$ descents.



Example 4 (Central factorial poset) We have

$$\sum_{n=0}^{\infty} U(n,k)x^n = \frac{x^k}{(1-1^2x)(1-2^2x)\cdots(1-k^2x)},$$

so that

$$\sum_{n=0}^{\infty} U(n+k,k)x^n = \frac{1}{(1-1^2x)(1-2^2x)\cdots(1-k^2x)}.$$

By the same kind of reasoning for Stirling posets, we see that $\Omega_P(n) = U(n+k,n)$.

Central factorial permutations

Theorem

$$\sum_{n\geq 0} U(n+k,n)t^n = \frac{\sum_{j=1}^{2k} d_{k,j} t^j}{(1-t)^{3k+1}},$$

where $d_{k,j}$ is the number of linear extensions of Central factorial poset with $j-1$ descents.

Central factorial permutations

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where $d_{k,j}$ is the number of linear extensions of Central factorial poset with $j-1$ descents.

Central factorial permutation: a permutation of the multiset $\{1,1,\overline{1},2,2,\overline{2},\ldots,n,n,\overline{n}\}$ with the following order

$$\overline{1} < 1 < \overline{2} < 2 \ldots < \overline{n} < n,$$

such that for each i, $1 \leq i \leq n$, the elements occurring between two occurrences of i are at least i.

Theorem (G-L-Z)

The integer $d_{k,j}$ *counts the number of Central factorial permutation of* $\{1, 1, \bar{1}, 2, 2, \bar{2}, \ldots, k, k, \bar{k}\}$ *with* $j-1$ *descents.*

Jacobi-Stirling polynomials

The Jacobi-Stirling polynomials (in n) $f_k(n;z)$ is defined by

$$f_k(n;z) := \mathrm{JS}(n+k, n; z),$$

which can be written as

$$f_k(n;z) = p_{k,0}(n) + p_{k,1}(n)z + \cdots + p_{k,k}(n)z^k.$$

Note: $p_{k,0}(n) = U(n+k,n)$ and $p_{k,k}(n) = S(n+k,n)$.

Theorem (G-L-Z)

For each integer k and i such that $0 \le i \le k$, there are positive integers $a_{k,i,j}$ for $1 \le j \le 2k-i$ such that

$$\sum_{n\ge 0} p_{k,i}(n)t^n = \frac{\sum_{j=1}^{2k-i} a_{k,i,j}t^j}{(1-t)^{3k-i+1}}.$$

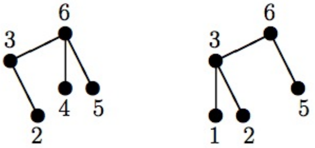
Jacobi-Stirling poset

1. Let R_k be the Central factorial poset on $[3k]$.
2. For any $S \subseteq [k]$, the Jacobi-Stirling poset $R_{k,S}$ is defined to be the poset obtained from R_k by removing the points $3m-2$ for $m \in S$.
3. For $0 \le i \le k$, the set $R_{k,i}$ is defined by

$$R_{k,i} = \{R_{k,S} \mid S \subseteq [k] \text{ with cardinality } i\}.$$

Example

$R_{2,1} = \{R_{2,\{1\}}, R_{2,\{2\}}\}$.



Jacobi-Stirling posets

Define $\mathscr{L}(R_{k,i})$ by

$$\mathscr{L}(R_{k,i}) = \bigcup_{\substack{S \subseteq [k] \\ |S|=i}} \mathscr{L}(R_{k,S}).$$

Theorem (G-L-Z)

The integer $a_{k,i,j}$ is the number of elements of $\mathscr{L}(R_{k,i})$ with $j-1$ descents, i.e.,

$$\sum_{\pi \in \mathscr{L}(R_{k,i})} t^{des(\pi)+1} = \sum_{j=1}^{2k-i} a_{k,i,j} t^j.$$

(Sketch of the proof) Using the generating function

$$\sum_{k\geq 0} f_k(n;z)t^k = \frac{1}{(1-(z+1)t)(1-2(z+2)t)\cdots(1-n(z+n)t)}$$

to show that

$$p_{k,i}(n) = \sum_{\substack{S\subseteq[k]\\|S|=i}} \Omega_{R_{k,S}}(n),$$

the theorem then follows from P-partition theory.

1. Let

$$M_n := \{1, 1, \overline{1}, 2, 2, \overline{2}, \dots, n, n, \overline{n}\}$$

with the following order

$$\overline{1} < 1 < \overline{2} < 2 \dots < \overline{n} < n.$$

2. For $0 \leq i \leq n$, denote by

$$M_{n,i} = \{M_n \setminus S \mid S \subseteq [\overline{n}] \text{ with cardinality } i\},$$

where $[\overline{n}] := \{\overline{1}, \overline{2}, \dots, \overline{n}\}$.

3. A Jacobi-Stirling permutation of $M_{n,i}$ is a permutation of a multiset on $M_{n,i}$ such that for each i, $1 \leq i \leq n$, the elements occurring between two occurrences of i are at least i.

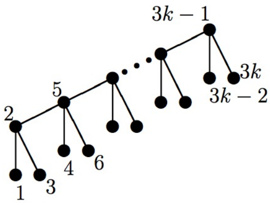
Theorem (G-L-Z)

The integer $a_{k,i,j}$ counts the number of Jacobi-Stirling permutations of $M_{k,i}$ with $j-1$ descents.

(Sketch of the proof) Let $\mathfrak{S}(M_{k,i})$ be the set of Jacobi-Stirling permutations of $M_{k,i}$. We construct a bijection

$$\phi_{k,i} : \mathscr{L}(R_{k,i}) \to \mathfrak{S}(M_{k,i})$$

by induction on k, which preserves the number of descents.



Example 3 (Legendre-Stirling poset)
$\Omega_P(n) = \mathrm{LS}(n-1+k, n-1)$.

Theorem (G-L-Z)

$$\sum_{n\geq 0} LS(n+k,n)t^n = \frac{\sum_{j=1}^{2k-1} b_{k,j} t^j}{(1-t)^{3k+1}},$$

where $b_{k,j}$ is the number of linear extensions of Legendre-Stirling poset with j descents.

Legendre-Stirling permutations

Definition

A Legendre-Stirling permutation is a Jacobi-Stirling permutation of the multiset $\{1, 1, \overline{1}, 2, 2, \overline{2}, \ldots, n, n, \overline{n}\}$ with the following order

$$\overline{1} = 1 < \overline{2} = 2 \ldots < \overline{n} = n.$$

Here $\overline{1} = 1$ means that neither $1\overline{1}$ nor $\overline{1}1$ counts as a descent.

Definition

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Here $\overline{1} = 1$ means that neither $1\overline{1}$ nor $\overline{1}1$ counts as a descent.
Note: $122\overline{2}1\overline{1}$, as a Legendre-Stirling permutation has 1 descent, while as a Jacobi-Stirling permutation has 3 descents.

Example

$1\overline{2}1\overline{1}22$ is a Legendre-Stirling permutations, while $2\overline{2}211\overline{1}$ is not.

Theorem (Egge, 2010)

The integer $b_{k,j}$ counts the number of Legendre-Stirling permutations of $\{1, 1, \overline{1}, 2, 2, \overline{2}, \ldots, k, k, \overline{k}\}$ with $j-1$ descents.

A q-analogue (with Ana F. Loureiro)

q-differential equations for q-classical polynomials and q-Jacobi-Stirling numbers

$$\big(D_q f\big)(x) := \frac{f(qx) - f(x)}{(q-1)x} \tag{1}$$

The q-classical polynomials share a number of properties and, among them, we single out the fact they are eigenfunctions of a second-order q-differential operator of q-Sturm-Liouville type,

$$\mathcal{L}_q := \Phi(x) D_q \circ D_{q^{-1}} - \Psi(x) D_{q^{-1}}$$

, where Φ is a monic polynomial of degree two at most and Ψ a polynomial of degree one.

The q-Jacobi-Stirling numbers

$$x^n = \sum_{k=0}^{n} \mathrm{JS}_n^k(z;q) \prod_{i=0}^{k-1} \left(x - [i]_q\left(z + [i]_{q^{-1}}\right)\right),$$

$$\prod_{i=0}^{n-1} \left(x - [i]_q\left(z + [i]_{q^{-1}}\right)\right) = \sum_{k=0}^{n} (-1)^{n-k} \mathrm{js}_n^{(k)}(z;q)\, x^k.$$

Thank you!