Polynomial Sequences of Binomial Type

Richard P. Stanley

M.I.T.

Some motivation

Let $D = \frac{d}{dn}$, acting on $f(n) \in \mathbb{C}[n]$. Then

$$\begin{aligned} Dn^k &= kn^{k-1} \\ f(n) &= \sum_{k \geq 0} D^k f(0) \frac{n^k}{k!} \text{ (Taylor series).} \end{aligned}$$

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$$

Let $\boldsymbol{\Delta} f(n) = f(n+1) - f(n)$ and $\boldsymbol{(n)_k} = n(n-1)\cdots(n-k+1)$. Then

$$
\begin{aligned}
\Delta(n)_k &= k(n)_{k-1} \\
f(n) &= \sum_{k\geq 0} \Delta^k f(0) \frac{(n)_k}{k!}.
\end{aligned}
$$

Connection between D and Δ

By Taylor's theorem,

$$f(n+x) = \sum_{k\geq 0} D^k f(n) \frac{x^k}{k!}.$$

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Put $x = 1$:

Connection (continued)

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$$\Rightarrow \Delta f(n) = (e^D - 1) f(n) \Rightarrow \boldsymbol{\Delta = e^D - 1}.$$

Thus also $D = \log(\Delta + 1)$.

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G.-C. Rota, *Finite Operator Calculus*, Academic Press, 1976.

The shift operator

Define $\boldsymbol{E}: \mathbb{C}[n] \to \mathbb{C}[n]$ by

$$Ef(n) = f(n+1).$$

Main thm. of operator calculus

Theorem. Let $L\colon \mathbb{C}[n] \to \mathbb{C}[n]$ be linear (over $\mathbb{C}$) and satisfy $L(n) = 1$ and $L(\deg d) = \deg d - 1$. The following two conditions are equivalent.

- $LE = EL$

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- $LE = EL$
- There exist polynomials $p_k(n)$, $k \geq 0$, such that $p_0(n) = 1$, $\deg p_k(n) = k$, and

$$\begin{aligned} Lp_k(n) &= kp_{k-1}(n) \\ \sum_{k\geq 0} p_k(n)\frac{x^k}{k!} &= \left(\sum_{k\geq 0} p_k(1)\frac{x^k}{k!}\right)^n . \end{aligned}$$

Binomial type

If $p_0(n), p_1(n), \ldots$ is a sequence of polynomials satisfying

$$\sum_{k\geq 0} p_k(n)\frac{x^k}{k!} = \left(\sum_{k\geq 0} p_k(1)\frac{x^k}{k!}\right)^n,$$

then we call $p_0(n), p_1(n), \ldots$ a **sequence of polynomials of binomial type**, or just **polynomials of binomial type**.

Further properties

If $LE = EL$ and $Lp_k(n) = kp_{k-1}(n)$, then:

- $$f(n) = \sum_{k \leq 0} L^k f(0) \frac{p_k(n)}{n!}$$
  (Taylor series analogue)
- L is a power series in D.
- $\ldots$

A characterization

Note. The condition $\deg p_k(n) = k$ is then equivalent to $p_1(n) \neq 0$ (or just $p_1(1) \neq 0$). Sometimes this extra condition is part of the definition of binomial type.

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Theorem. *A sequence* $p_0(n) = 1, p_1(n), \ldots$ *of polynomials is of binomial type if and only if*

$$p_k(m+n) = \sum_{i=0}^{k} \binom{k}{i} p_i(m) p_{k-i}(n), \ k \geq 0.$$

Some classical examples

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$$\sum_{k\geq 0} \boldsymbol{(n)_k} \frac{x^k}{k!} = \sum_{k\geq 0} \binom{n}{k} x^k = (1+x)^n$$

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More on Abel polynomials

Binomial type is equivalent to **Abel's identity**:

$$(x+y)^k = \sum_{i=0}^{k} \binom{k}{i} x(x-iz)^{i-1}(y+iz)^{k-i}.$$

Note that $z = 0$ gives the binomial theorem.

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Closely related to tree enumeration.

Yet another example

- $p_k(n) = \sum_{i=1}^{k} \underbrace{\boldsymbol{S(k,i)}}_{\textbf{Stirling no. of 2nd kind}} n^i$

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$$\sum_{k\geq 0} \left(\sum_i S(k,i) n^i\right) \frac{x^k}{k!} = \left(\sum_{k\geq 0} \underbrace{B(k)}_{\text{Bell number}} \frac{x^k}{k!}\right)^n$$

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$$\sum_{k\geq 0} \left(\sum_{i} \binom{k}{i} i^{k-i} n^i \right) \frac{x^k}{k!} = \exp nxe^x$$

More examples?

Are there interesting examples of polynomials of binomial type for which explicit formulas don't exist?

Binomial posets

$\boldsymbol{P} = P_0 \cup P_1 \cup \cdots$ (disjoint union): a poset (partially ordered set) such that all maximal chains have the form $t_0 < t_1 < \ldots$, where $t_i \in P_i$.

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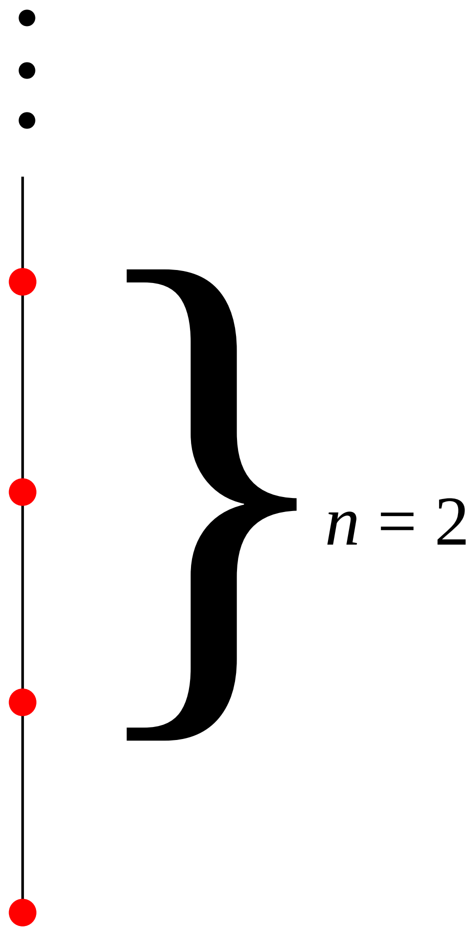
Write **rank**$(t_i) = i$. Then P is a **binomial poset** if for all $s \leq t$, where $\boldsymbol{k} = \mathrm{rank}(t) - \mathrm{rank}(s)$, the number of (saturated) chains $s = t_0 < t_1 < \cdots < t_k = t$ depends only on k.

Call this number $\boldsymbol{B(k)}$ (**factorial function** of P).

Chains

- $P = \{0, 1, 2, \ldots\}$ (a chain): $B(k) = 1$.



Two further examples

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- $P = \mathbb{B}(\boldsymbol{q})$, the set of all finite-dimensional subspaces of an infinite-dimensional vector space over the finite field $\mathbb{F}_q$:

$$B(k) = \boldsymbol{(k)!} = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{k-1}).$$

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$\mathbb{B}(q)$ is a **q-analogue** of $\mathbb{B}$.

Multichains

Theorem. *Let P be a binomial poset. Let $p_k(n)$ be the number of multichains*

$$s = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n = t,$$

where $\mathrm{rank}(t) - \mathrm{rank}(s) = k$. *Then*

$$\sum_{k \geq 0} p_k(n) \frac{x^k}{B(k)} = \left(\sum_{k \geq 0} \frac{x^k}{B(k)} \right)^n .$$

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Corollary. $k!\, p_k(n)/B(k)$, $k \geq 0$, *is a sequence of polynomials of binomial type.*

Example: $\mathbb{B}(q)$

Let $r_k(n)$ be the number of multichains of subspaces

$$\{0\} = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \mathbb{F}_q^k.$$

Let

$$p_k(n) = \frac{k!\, r_k(n)}{(\boldsymbol{k})!}.$$

Then $p_0(n), p_1(n), \ldots$ is a sequence of polynomials of binomial type.

Other example

Many other examples of binomial posets, e.g.,

$$B(k) = \frac{1}{2^{\binom{k}{2}} k!}.$$

Related to graph colorings and acyclic orientations.

New vistas: toroidal graphs

$\mathbb{Z}_{\boldsymbol{n}}^{\boldsymbol{d}}$: the $n \times n \times \cdots \times n$ (d times) d-dimensional toroidal graph.

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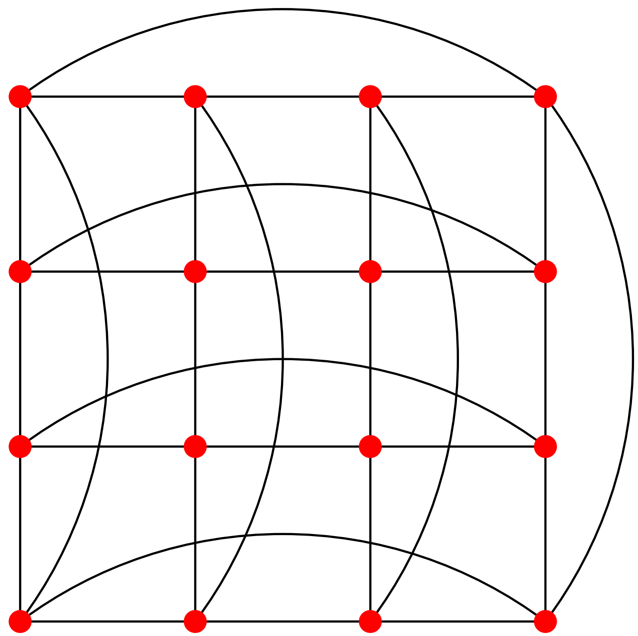

$\mathbb{Z}_4^2$

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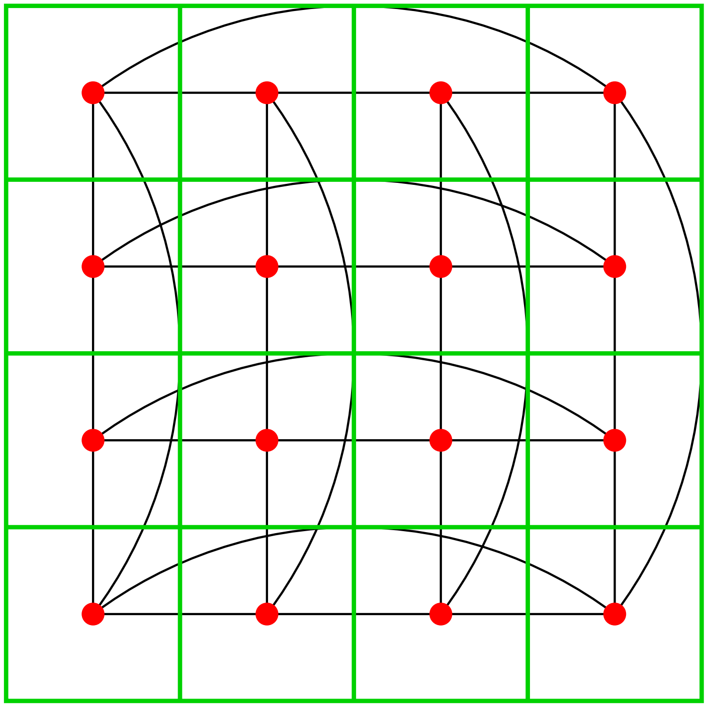
$\mathbb{Z}_{\boldsymbol{n}}^{\boldsymbol{d}}$: the $n \times n \times \cdots \times n$ (d times) d-dimensional toroidal graph.



The green squares are the vertices of $\mathbb{Z}_4^2$.

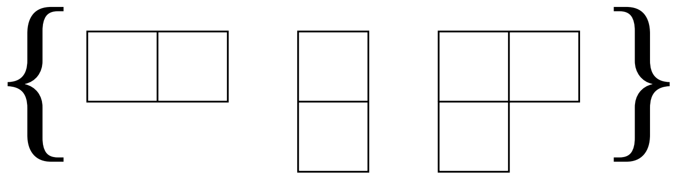
Algebraic definition

$\mathbb{Z}_{\boldsymbol{n}}$: integers modulo n

$\mathbb{Z}_{\boldsymbol{n}}^{\boldsymbol{d}}$: $\{(a_1, \ldots, a_d) \,:\, a_i \in \mathbb{Z}_n\}$ (vertex set)

$\alpha = (a_1, \ldots, a_d)$ and $\beta = (b_1, \ldots, b_d)$ are **adjacent** if $\alpha - \beta$ has one nonzero coordinate, which is equal to ± 1 (modulo n).

Figures

a set S of **figures**: 

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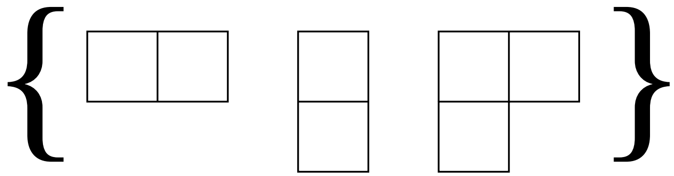
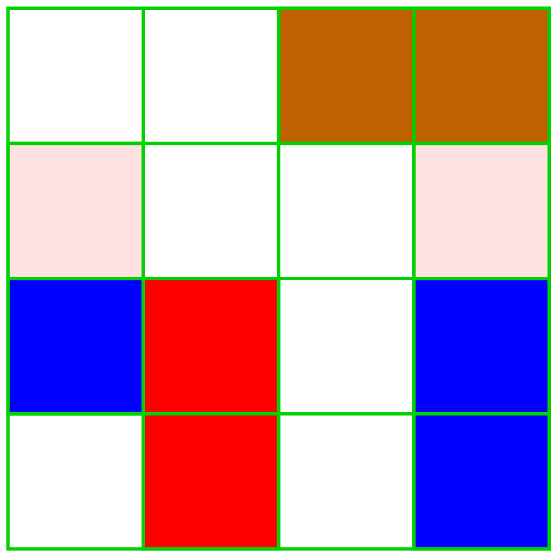
a set S of **figures**: 

A **placement** of S on $\mathbb{Z}_4^2$:



The function $f_k(n^d)$

Fix d and a finite set S of tiles.

$\boldsymbol{f_k(n^d)}$: number of placements of S on $\mathbb{Z}_n^d$ covering a total of k $1 \times 1 \times \cdots \times 1$ boxes.

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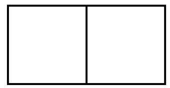
Example. $S = \{\square\}$. Then $f_k(n^2) = \binom{n^2}{k}$

Another example

Example. $S = \{$  $\}$

$$f_{2j+1}(n^2) = 0$$
$$f_2(n^2) = n^2$$
$$f_4(n^2) = \frac{1}{2}n^2(n^2-3)$$

Still another example

Example. $S = \{\square \quad \square\!\square\}$

$$
\begin{aligned}
f_1(n^2) &= n^2 \\
f_2(n^2) &= \binom{n^2}{2} + n^2 \\
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\end{aligned}
$$

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Note that these are **polynomials** in n^2.

Relationship to binomial type

Theorem (Jon Schneider)

(a) *For $n \gg 0$ (so all tiles fit on $\mathbb{Z}_n^d$), there is a polynomial p_k for which $p_k(n) = f_k(n^d)$.*

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(a) *For* $n \gg 0$ *(so all tiles fit on* $\mathbb{Z}_n^d$*), there is a polynomial* p_k *for which* $p_k(n) = f_k(n^d)$.

(b) $p_0, 1!\,p_1, 2!\,p_2, \ldots$ *is a sequence of polynomials of binomial type.*

An example

Recall: $S = \{ \square \quad \square\!\square \}$

$$f_1(n^2) = n^2, \quad f_2(n^2) = \binom{n^2}{2} + n^2$$

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$$1 + nx + \left(\binom{n}{2} + n\right)x^2 + \left(\binom{n}{3} + n(n-2)\right)x^3$$

$$+ \cdots = (1 + x + x^2 - x^3 + \cdots)^n$$

Chromatic polynomials

G: finite graph with vertex set V, $q \geq 1$

$\chi_G(q)$: number of **proper** colorings

$$f : V \to \{1, \ldots, q\},$$

i.e., adjacent vertices get different colors

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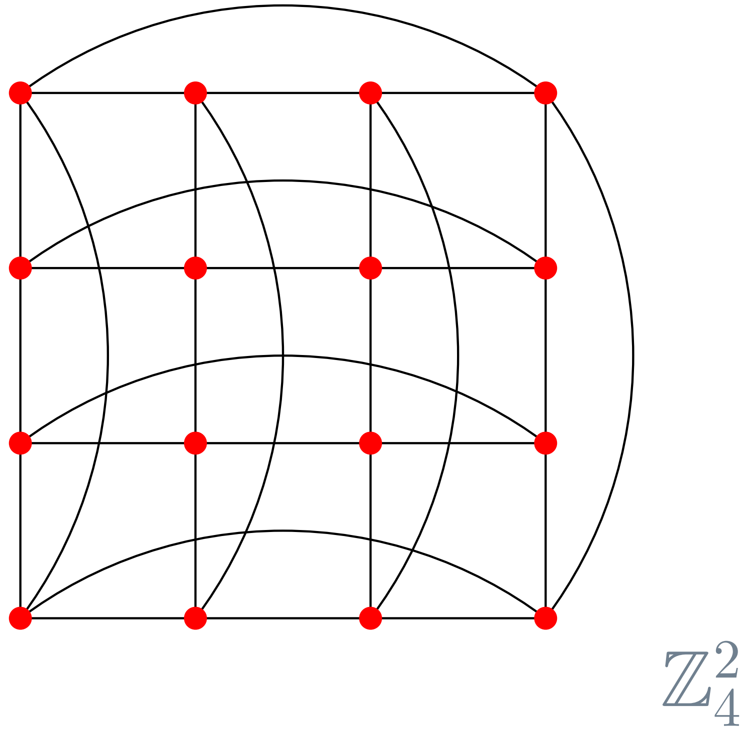
Example. $G = K_n$, complete graph with n vertices. Then

$$\chi_{K_n}(q) = q(q-1)\cdots(q-n+1).$$

The graph $\mathbb{Z}_n^d$

Recall $\mathbb{Z}_n^d$ is a graph:



Much interest from physicists in the chromatic polynomial $\chi_{\mathbb{Z}_n^d}(q)$.

A trivial and nontrivial result

$$\chi_{\mathbb{Z}_n^d}(2) = \begin{cases} 2, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

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Theorem (**E. Lieb**, 1967)

$$\lim_{n\to\infty} \chi_{\mathbb{Z}_n^2}(3)^{1/n^2} = \left(\frac{4}{3}\right)^{3/2} = 1.5396\cdots$$

(**residual entropy of square ice**)

Open variants

$$\lim_{n\to\infty} \chi_{\mathbb{Z}_n^2}(3)^{1/n^2} = \left(\frac{4}{3}\right)^{3/2}$$

$\lim_{n\to\infty} \chi_{\mathbb{Z}_n^2}(4)^{1/n^2}$: not known

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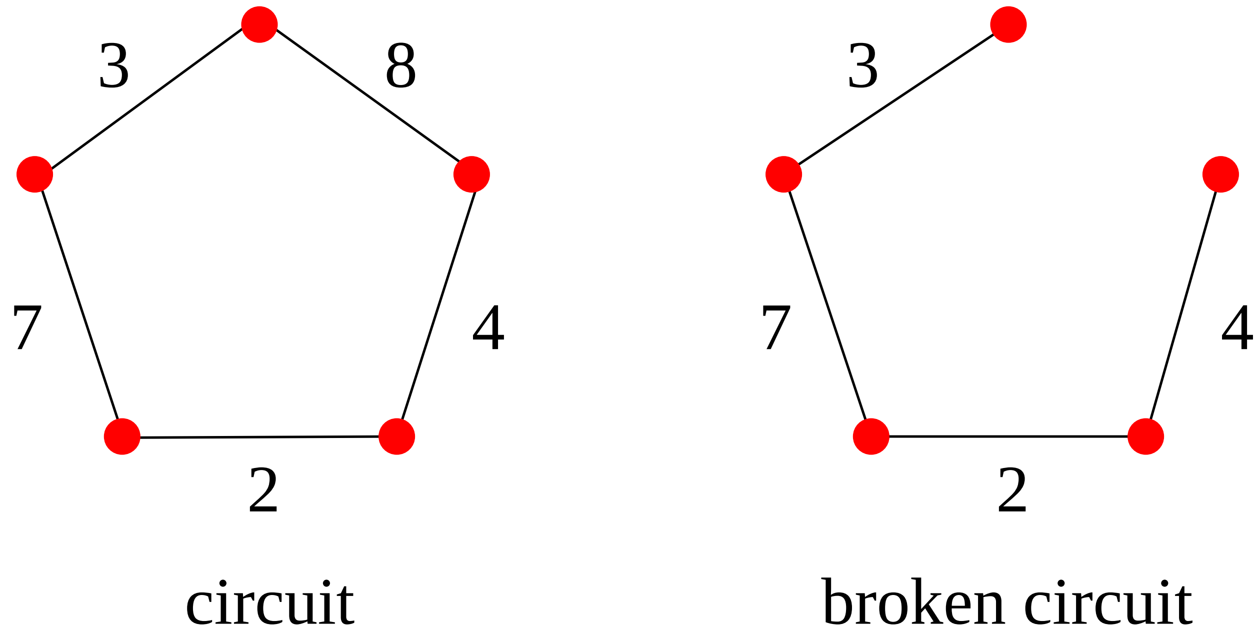
Broken circuits

Label the edges of the graph G as $1, 2, \ldots, m$.

broken circuit: a circuit with its largest edge removed



The broken circuit theorem

Theorem (**H. Whitney**, 1932) *Let G have N vertices. Write*

$$\chi_G(q) = a_0 q^N - a_1 q^{N-1} + a_2 q^{N-2} - \cdots .$$

Then a_i is the number of i-element sets of edges of G that contain no broken circuit.

An example

Example. If G is a 4-cycle, then no 0-element, 1-element, or 2-element set of edges contains a broken circuit. One 3-element set contains (in fact, is) a broken circuit, and all four edges contain a broken circuit. Hence

$$\begin{aligned} \chi_G(q) &= q^4 - \binom{4}{1} q^3 + \binom{4}{2} q^2 - \left(\binom{4}{3} - 1 \right) q \\ &= q^4 - 4q^3 + 6q^2 - 3. \end{aligned}$$

Broken circuits in $\mathbb{Z}_n^2$

Let $\boldsymbol{G} = \mathbb{Z}_n^2$ and $\boldsymbol{N} = n^2$ (number of vertices), so $2N$ edges. The smallest cycle in G has length four. There are N such cycles, so N 3-element sets of edges containing (in fact, equal to) a broken circuit. Hence

$$\begin{aligned}\chi_{\mathbb{Z}_n^2}(q) &= q^N - \binom{2N}{1} q^{N-1} + \binom{2N}{2} q^{N-2} \\ &\quad - \left(\binom{2N}{3} - N\right) q^{N-3} + \cdots\end{aligned}$$

Chromatic polynomial of $\mathbb{Z}_n^d$

Theorem (**J. Schneider**). *Let* $N = n^d$*, the number of vertices of* $\mathbb{Z}_n^d$*. Write*

$$\chi_{\mathbb{Z}_n^d}(q) = c_0(N)q^N - c_1(N)q^{N-1} + c_2(N)q^{N-2} - \cdots .$$

Then for $N \gg 0$*,* $c_k(N)$ *agrees with a polynomial* $p_k(N)$*. Moreover,* $p_0, 1!\, p_1, 2!\, p_2, \ldots$ *is a sequence of polynomials of binomial type.*

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Proof uses a variant of Schneider's previous result on placing tiles on $\mathbb{Z}_n^d$.

Computations

Let $d = 2$. **D. Kim** and **I. G. Enting** made a computation (1979) equivalent to

$$\sum_{k\geq 0} p_k(N)x^k = (1 + 2x + x^2 - x^3 + x^4 - x^5 + x^6 \\ -2x^7 + 9x^8 - 38x^9 + 130x^{10} \\ -378x^{11} + 987x^{12} - 2436x^{13} \\ +5927x^{14} - 14438x^{15} + 34359x^{16} \\ -75058x^{17} + 134146x^{18} + \cdots)^N.$$

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Can anything be said about these numbers?

Does the series converge for small x?

Some small values

$$
\begin{aligned}
p_1(N) &= 2N \\
p_2(N) &= 2N(2N-1) \\
p_3(N) &= 2N(4N^2-6N-1) \\
p_4(N) &= 4N(N+1)(2N-3)(2N-5) \\
p_5(N) &= 8N(N+2)(N-2)(2N-3)(2N-7) \\
p_6(N) &= 8N(8N^5-60N^4+50N^3+495N^2 \\
&\quad -1228N+825) \\
p_7(N) &= 8N(16N^6-168N^5+280N^4+2310N^3 \\
&\quad -10241N^2+14553N-8010)
\end{aligned}
$$

Further directions

What about $n_1 \times n_2 \times \cdots \times n_d$ tori?

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Nothing new: simply replace $N = n^d$ with $N = n_1 n_2 \cdots n_d$.

Tutte polynomials

What about replacing chromatic polynomials with Tutte polynomials?

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Currently under investigation. No known satisfactory generalization of broken circuit theorem.

Multi-indexed polynomials

What about replacing $p_k(n)$ with $p_{j,k}(n)$?

Multi-indexed polynomials

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To be investigated.

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Interesting example. $\boldsymbol{K_{jk}}$: complete bipartite graph

Theorem (EC2, Exercise 5.6).

$$\sum_{j,k\geq 0} \chi_{K_{jk}}(n) \frac{x^j}{j!} \frac{y^k}{k!} = (e^x + e^y - 1)^n$$

Multivariate polynomials

What about replacing $p_k(n)$ with $p_k(m, n)$?

Multivariate polynomials

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Not yet considered.

May involve $F(x)^m G(x)^n$.

Continuous variant

Theorem (**Schneider**). *Let $\mathcal{S}$ be a bounded measurable set in d-dimensional Euclidean space. Let $P_k(n^d)$ be the probability that no two copies intersect when we place k copies of $\mathcal{S}$ independently and uniformly at random inside a d-dimensional torus of side length n. Then $n^{dk}P_k(n^d)$ is eventually a polynomial $p_k(n)$ for each k, and these polynomials form a sequence of binomial type.*

A reference

arXiv:1206.6174

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