The Combinatorics of Two Dimensional Hermite Polynomials and Their q-Analogues

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Two types of complex Hermite polynomials. First type is simply the Hermite polynomials in the complex variable z, that is $\{H_n(z)\}$. These polynomials have been introduced in the study of coherent states They are

$$H_n(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!\,(-1)^k}{k!\,(n-2k)!}(2z)^{n-2k}, \qquad z = x+iy.$$

The second type is the polynomials $\{H_{m,n}(z_1, z_2)\}$

$$H_{m,n}(z_1, z_2) = \sum_{k=0}^{m \wedge n} (-1)^k k! \binom{m}{k}\binom{n}{k} z_1^{m-k} z_2^{n-k}.$$

If m or $n = 0$ you get z_2^n or z_1^m.

Their exponential generating function is

$$\sum_{m,\,n=0}^{\infty} H_{m,n}(z_1, z_2)\frac{u^m\,v^n}{m!\,n!} = e^{uz_1+vz_2-uv}.$$

The complex Hermite polynomials $\{H_n(z)\}$ satisfy the orthogonality relations, with $z = x+iy$,

$$\int_{\mathbb{R}^2} H_{m,n}(z,\bar{z})\overline{H_{r,s}(z,\bar{z})}e^{-x^2-y^2}\,dx\,dy$$

$$= \pi m!\,n!\,\delta_{m,r}\,\delta_{n,s}.$$

$$\int_{\mathbb{R}^2} H_m(x+iy)\overline{H_n(x+iy)}\,e^{-ax^2-by^2}\,dx\,dy$$

$$= \frac{\pi}{\sqrt{ab}}\,2^n n!\left(\frac{a+b}{ab}\right)^n \delta_{m,n}$$

where

$$0 < a < b, \qquad \frac{1}{a} = 1 + \frac{1}{b}.$$

Let k be a fixed positive integer and let $\mathbf{n} = (n_1, \ldots, n_k)$ be a k-tuple of nonnegative integers, that is $\mathbf{n} \in \mathbb{N}_0^k$. Set $|\mathbf{n}| = \sum_{j=1}^k n_j$. In 1982 Azor, Gillis, and Victor, and Godsil, independently, found a combinatorial interpretation of the integrals

$$A(\mathbf{n}) = \frac{1}{\sqrt{\pi}}\,2^{-\frac{1}{2}|\mathbf{n}|}\int_{\mathbb{R}} e^{-x^2}\prod_{j=1}^{k} H_{n_j}(x)\,dx.$$

They count the number of inhomogeneous perfect matchings of a multiset with k sets (components) of sizes (number of elements) $n_1, \dots, n_k$. Azor-Gillis-Victor and Godsil.

Asymptotics. Typical

$$\int_{\mathbb{R}} \prod_{j=1}^{m} p_{n_j}(\lambda_j x) w(x) dx.$$

The polynomials satisfy the three term recurrence relations

$$\begin{aligned} zH_{r,s}(z,\overline{z}) &= sH_{r,s-1}(z,\overline{z}) + H_{r+1,s}(z,\overline{z}) \\ \overline{z}H_{r,s}(z,\overline{z}) &= rH_{r-1,s}(z,\overline{z}) + H_{p,s+1}(z,\overline{z}). \end{aligned}$$

When $m = n + s \geq n$ then

$$H_{n+s,n}(z,\bar{z}) = (-1)^n n!\, z^s L_n^{(s)}(|z|^2).$$

Some Names:

Initiated: K. Ito, (1952),

S. T. Ali, F. Bagarello, and G. Honnouvo, (2010),

N. Cotfas, J. P. Gazeau, and K. Górska, (2010),

A. Ghanmi, (2008) and A. Ghanmi, (2013),

A. Intissar and A. Intissar, (2006),

Shigekawa, (1987),

K. Thirulogasanthar, G. Honnouvo, and A. Krzyzak, (2010),

A. Wünsche, (1998) and (1999).

The Combinatorics: All the combinatorics involving $H_{m,n}$ are from a joint work with Plamen Simeonov, Proc. AMS, to appear.

A Combinatorial Problem: Suppose that we have two colored multisets. The first, of color I, has size $\mathbf{m} = (m_1, m_2, \cdots, m_k)$ and the second, of color II, has size $\mathbf{n} = (n_1, n_2, \cdots, n_k)$. We connect objects from different sets by edges. An edge has weight 1 if its vertices have the same color and has weight $1/a + 1/b$ if its vertices have different colors. The weight of a matching is the product of the weights of its edges.

Theorem 1. *The total weight of all perfect matchings of this type is the number $B(\mathbf{m}, \mathbf{n})$ defined by*

$$B(\mathbf{m}, \mathbf{n}) = \frac{\sqrt{ab}}{\pi} 2^{-\frac{1}{2}(|\mathbf{m}|+|\mathbf{n}|)}$$

$$\times \int_{\mathbb{R}^2} \prod_{j=1}^{k} [H_{m_j}(z) \overline{H_{n_j}(z)}] \, e^{-ax^2-by^2} \, dx \, dy.$$

One orthogonality relations follow from the Theorem.

Note that $H_{m,n}$ has real coefficients.

Theorem 2. *Suppose that we have two color I multisets B_1 and B_2 and two color II multisets R_1 and R_2, each having k components. Multisets B_1 and B_2 have sizes $\mathbf{m}$ and $\mathbf{p} \in \mathbb{N}_0^k$ and multisets R_1 and R_2 have sizes $\mathbf{n}$ and $\mathbf{q} \in \mathbb{N}_0^k$, respectively. Consider all inhomogeneous perfect matching of these four multisets where each element is matched to an element of different color. Elements from same index components of B_1 and R_1 can not match each other, and the same restriction holds for B_2 and R_2. Then the number of perfect matchings of this type $I(\mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{q})$ is*

$$\frac{1}{\pi} \int_{\mathbb{R}^2} \prod_{j=1}^{k} H_{m_j,n_j}(z, \bar{z}) H_{p_j,q_j}(z, \bar{z})\, e^{-x^2-y^2}\, dx\, dy.$$

Exponential generating function of the numbers $\{B(\mathbf{m},\mathbf{n})\}$.

Theorem 3. *Suppose that* a *and* b *satisfy* $1/a = 1 + 1/b$. *Then*

$$\sum_{m_1,\ldots,m_k,\,n_1,\ldots,n_k=0}^{\infty} B(\mathbf{m},\mathbf{n}) \prod_{j=1}^{k} \frac{s_j^{m_j} t_j^{n_j}}{m_j! n_j!}$$

$$= \exp\left[\sum_{1\le i<j\le k} (s_i s_j + t_i t_j) + \frac{a+b}{ab} \sum_{1\le i,j\le k} s_i t_j \right].$$

Define the numbers

$$B(\mathbf{n}_1, \ldots, \mathbf{n}_s) = \frac{2^{-\alpha_s \sum_{\nu=1}^{s} |\mathbf{n}_\nu|} \prod_{l=1}^{s} a_l^{1/2}}{\pi^{s/2}}$$

$$\times \int_{\mathbb{R}^s} \prod_{\nu=1}^{s} \prod_{j=1}^{k} H_{n_{\nu,j}}\left(L_\nu(\vec{x})\right) e^{-\sum_{l=1}^{s} a_l x_l^2}\, dx_1 \cdots dx_s$$

where $\mathbf{n}_\nu = (n_{\nu,1}, \ldots, n_{\nu,k}) \in \mathbb{N}_0^k$, $\nu = 1, \ldots, s$, $\alpha_s \geq 0$, $a_l > 0$, $l = 1, \ldots, s$, $\vec{x} = (x_1, \ldots, x_s)$, and

$$L_\nu(\vec{x}) = \sum_{l=1}^{s} \omega_{\nu,l} x_l, \qquad \nu = 1, \ldots, s$$

are linear functions with complex coefficients satisfying the conditions

$$\sum_{l=1}^{s} \frac{\omega_{\nu,l}^2}{a_l} = 1, \qquad \nu = 1, \ldots, s.$$

The case $s = 2$ and $k = 1$ is worth recording separately. In this case, with $\alpha_s = \alpha_2 = 1/2$ we obtain the orthogonality relation

$$\int_{\mathbb{R}^2} H_m(\alpha x + \beta y) H_n(\gamma x + \delta y) e^{-ax^2 - by^2}\, dx\, dy$$
$$= \frac{\pi}{\sqrt{ab}} 2^n n! \left(\frac{\alpha\gamma}{a} + \frac{\beta\delta}{b} \right)^n \delta_{m,n}$$

where

$$a > 0, \quad b > 0, \qquad \frac{\alpha^2}{a} + \frac{\beta^2}{b} = \frac{\gamma^2}{a} + \frac{\delta^2}{b} = 1.$$

The orthogonality relations are special cases.

Theorem 4. *The numbers defined above have the generating function*

$$\sum_{\substack{n_{\nu,j}=0 \\ 1 \le \nu \le s,\, 1 \le j \le k}}^{\infty} B(\mathbf{n}_1, \ldots, \mathbf{n}_s) \prod_{\nu=1}^{s} \prod_{j=1}^{k} \frac{t_{\nu,j}^{n_{\nu,j}}}{n_{\nu,j}!}$$
$$= \text{exponential of}$$
$$2^{-2\alpha_s} \sum_{\substack{(\nu_1,j_1) \neq (\nu_2,j_2) \\ 1 \le \nu_1, \nu_2 \le s,\, 1 \le j_1, j_2 \le k}} \left[\sum_{l=1}^{s} \frac{\omega_{\nu_1,l}\omega_{\nu_2,l}}{a_l} \right] t_{\nu_1,j_1} t_{\nu_2,j_2}.$$

Let $\{p_n(x)\}$ be a sequence of orthogonal polynomials

$$\int_{\mathbb{R}} p_m(x)p_n(x)d\mu(x) = \zeta_n\delta_{m,n}, \quad \zeta_0 = 1.$$

The condition $\zeta_0 = 1$ is a normalization. The polynomials $\{p_n(x)\}$ must satisfy a three term recurrence relation of the form

$$p_{n+1}(x) = [A_n x + B_n]\, p_n(x) - C_n p_{n-1}(x),$$

for $n > 0$, and we will always assume $p_0(x) := 1, p_1(x) = A_0 x + B_0$. Therefore

$$\zeta_n = \frac{A_0}{A_n} C_1 C_2 \cdots C_n.$$

Consider the linearization coefficients in the expansion of $\prod_{j=1}^{m-1} p_{n_j}(\lambda_j x)$ in $\{p_n(x)\}$. Equivalently we consider the numbers

$$I(\mathbf{n}) := \int_{\mathbb{R}} \prod_{j=1}^{m} p_{n_j}(\lambda_j x)\, d\mu(x),$$

where $\mathbf{n} = (n_1, \ldots, n_m)$, n_j is a nonnegative integer for $1 \le j \le m$.

Notation

$$I_j^{\pm}(\mathbf{n}) = I(n_1, \ldots, n_{j-1}, n_j \pm 1, n_{j+1}, \ldots, n_m).$$

Moreover we assume that $\lambda_m = 1$. It is clear that we have the boundary condition

$$I_j^{+}((0, \ldots, 0, n_m))$$

$$= \lambda_j C_1 \frac{A_0}{A_1}\, \delta_{n_m,1} + B_0(1 - \lambda_j)\, \delta_{n_m,0}, \text{if} \quad n_m = 0, 1,$$

$$I((0, \ldots, 0)) = 1, \quad I(\mathbf{n}) = 0, \text{if} \sum_{j=1}^{m-1} n_j < n_m.$$

Separation of Variables:

Theorem 5. *The numbers $I(\mathbf{n})$ satisfy the system of difference equations*

$$I_j^+(\mathbf{n}) - \frac{A_{n_j}\lambda_j}{A_{n_k}\lambda_k} I_k^+(\mathbf{n}) = \left[B_{n_j} - \frac{A_{n_j}\lambda_j}{A_{n_k}\lambda_k} B_{n_k}\right] I(\mathbf{n})$$
$$-C_{n_j} I_j^-(\mathbf{n}) + \frac{A_{n_j}\lambda_j}{A_{n_k}\lambda_k} C_{n_k} I_k^-(\mathbf{n}).$$

Theorem 6. *The system of equations and the boundary conditions have a unique solution and is given by the integrals.*

Theorems 5 and 6 are due to Ismail-Kasraoui-Zeng 2013.

Let $K(\mathbf{n})$ be the number of inhomogeneous matchings of $[\mathbf{n}]$.

Theorem 7. *For $k, j \in [m]$ and $k \neq j$ the numbers $K(\mathbf{n})$ satisfy*

$$K_j^+(\mathbf{n}) - K_k^+(\mathbf{n}) = n_k K_k^-(\mathbf{n}) - n_j K_j^-(\mathbf{n}),$$

and the boundary condition with $\lambda_j = 1$ for all j and $A_0 C_1 = A_1$.

Let $r \in [m]$ and set $N_r = n_1 + \cdots + n_r$. For any $i \neq r$, the number of matchings in $\mathcal{K}_r^+(\mathbf{n})$ in which $N_r + 1$ is matched with an element in S_i is clearly $n_i K_i^-(\ltimes)$. This implies that for any $r \in [m]$, we have

$$K_r^+(\mathbf{n}) = \sum_{\substack{i=1 \\ i \neq r}}^{m} n_i K_i^-(\mathbf{n}),$$

which implies the result. The boundary conditions in are satisfied.

Work in Progress:
1. Two variable extension.
2. Asymptotics.

Derangements:

$$D(n_1, n_2, \cdots, n_m) = (-1)^{\sum_{j=1}^m} \int_0^\infty e^{-x} \prod_{j=1}^{m} L_{n_j}(x) dx.$$

It is known that

$$\lim_{m \to \infty} \frac{(n!)^m}{(mn)!} D(n, n, \cdots, n) = e^{-n}.$$

q-**Analogues**

Notation

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}.$$

The first q-analogue of $\{H_{m,n}(z_1,z_2)\}$ is defined by

$$H_{m,n}(z_1,z_2|q)$$
$$= \sum_{k=0}^{m\wedge n} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} (q;q)_k z_1^{m-k} z_2^{n-k}.$$

The polynomials $\{H_{m,n}(z_1,z_2|q)\}$ have the generating function

$$\sum_{n=0}^{\infty} H_{m,n}(z_1,z_2|q) \frac{u^m\ v^n}{(q;q)_m(q;q)_n} = \frac{(uv;q)_\infty}{(uz_1;q)_\infty(vz_2;q)_\infty}$$

They satisfy the recurrence relations

$$z_1 H_{m,n}(z_1,z_2|q)$$
$$= q^m(1-q^n)H_{m,n-1}(z_1,z_2|q) + H_{m+1,n}(z_1,z_2|q),$$
$$z_2 H_{m,n}(z_1,z_2|q)$$
$$= q^n(1-q^m)H_{m-1,n}(z_1,z_2|q) + H_{m,n+1}(z_1,z_2|q)$$

$$H_{m,n}(z_1, z_2|q)$$
$$= (-1)^n \frac{(q;q)_m \, q^{\binom{n}{2}}}{(q;q)_{m-n}} z_1^{m-n} p_n\left(z_1 z_2, q^{m-n}|q\right),$$

where $p_n(x; q^\alpha|q)$ is the little q-Laguerre or Wall's polynomials,

$$p_n(x; a|q) = {}_2\phi_1\left(\begin{array}{c} q^{-n}, 0 \\ aq \end{array} \middle| q; qx \right).$$

The polynomials $\{H_{m,n}(z,\overline{z}|q)\}$ satisfy the following orthogonality

$$\int_{\mathbb{C}} H_{m,n}(z,\overline{z}|q)\,\overline{H_{s,t}(z,\overline{z}|q)}d\mu(z,\overline{z})$$
$$= \frac{q^{mn}(q;q)_m(q;q)_n}{(q;q)_\infty}\delta_{m,s}\delta_{n,t},$$

where

$$d\mu(z,\overline{z}) = \frac{d\theta}{2\pi} \otimes \sum_{k=0}^{\infty} \frac{q^k}{(q;q)_k}\delta\left(r - q^{k/2}\right),$$

and $z = re^{i\theta}, r \in \mathbb{R}^+, \theta \in [0, 2\pi],\ m,n,s,t \in \mathbb{N}_0$.

The second q-analogue is

$$h_{m,n}(z_1, z_2; q)$$
$$= \sum_{j=0}^{m\wedge n} \begin{bmatrix} m \\ j \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q q^{(m-j)(n-j)} (-1)^j (q;q)_j \, z_1^{m-j} z_2^{n-j}.$$

Therefore

$$h_{m,n}(z_1, z_2; q) = q^{mn} i^{m+n} H_{m,n}(iz_1, iz_2 | 1/q)$$

The polynomials $\{h_{m,n}(z_1, z_2; q)\}$ have the generating function

$$\sum_{m,n=0}^{\infty} \frac{h_{m,n}(z_1, z_2; q)}{(q;q)_m (q;q)_n} q^{(m-n)^2/2} u^m v^n$$
$$= \frac{\left(-q^{1/2} u z_1; q)_\infty (-q^{1/2} v z_2; q\right)_\infty}{(-uv;q)_\infty},$$

They satisfy the recursions

$$z_1 h_{m,n}(z_1, z_2; q) = h_{m+1,n}(z_1, z_2; q)$$
$$+ (1 - q^n) q^{-m-n} h_{m,n-1}(z_1, z_2; q),$$
$$z_2 h_{m,n}(z_1, z_2; q) = h_{m,n+1}(z_1, z_2; q)$$
$$+ q^{-m-n} (1 - q^m) h_{m-1,n}(z_1, z_2; q).$$

Moreover we note the relation

$$h_{m,n}(z_1, z_2; q) = (-1)^n (q;q)_n z_1^{m-n} L_n^{(m-n)}(z_1 z_2; q),$$

where $\{L_n^{\alpha}(z)\}$ are q-Laguerre polynomials.

Theorem 8. *The polynomials* $\{h_{m,n}(z, \overline{z}; q)\}$ *satisfy the following orthogonality*

$$\int_{\mathbb{R}^2} h_{m,n}(z, \overline{z}; q) \overline{h_{s,t}(z, \overline{z}; q)} \frac{dxdy}{(-z\overline{z}; q)_\infty} = \frac{\pi \log q^{-1} (q;q)_m (q;q)_n}{q^{(m-n)^2/2+(m+n)/2}} \delta_{m,s}\delta_{n,t},$$

where $z = x + iy$, *and* $m, n, s, t \in \mathbb{N}_0$.

The orthogonality measure of the polynomials $L_n^{(\alpha)}(x; q)$ is not unique. If μ is a normalized orthogonality measure then

$$d\sigma\left(re^{i\theta}, re^{-i\theta}\right) = \frac{1}{2} d\theta d\mu(r^2), \quad r \in \mathbb{R}^+, \theta \in [0, 2\pi]$$

is also an orthogonal measure for $h_{m,n}\left(re^{i\theta}, re^{-i\theta}; q\right)$ where $r \in \mathbb{R}^+, \theta \in [0, 2\pi]$.

Interpretation for q-Hermite polynomials. PM is the set of perfect matchings. For $\pi \in PM$, $c(\pi)$ is the crossing number of π.

$$\int_{-1}^{1} \prod_{j=1}^{k} h_{n_j}(x;q) w(x) dx = \sum_{\pi \in PM} q^{c(\pi)}.$$

Work in Progress

Linearization of products and their analysis. Rogers-Ramanujan type of identities.

$2D$-q-Ultraspherical Polynomials.

These are q-analogues of the disc polynomials. For $0<q<1$ and $qb<1$, let

$$p_{m,n}(z_1,z_2;b|q)$$
$$=\sum_{k=0}^{\infty}\begin{bmatrix}m\\k\end{bmatrix}_q\begin{bmatrix}n\\k\end{bmatrix}_q\frac{q^{\binom{k}{2}}(q;q)_k(bq;q)_{m+n-k}}{(-1)^k z_1^{k-m}z_2^{k-n}},$$

it is clear that

$$p_{m,n}(z_2,z_1;b|q)=p_{n,m}(z_1,z_2;b|q)$$

For $m\ge n$ we have

$$p_{m,n}(z_1,z_2;b|q)=(-1)^n q^{\binom{n}{2}}$$
$$\times(bq;q)_m\left(q^{m-n+1};q\right)_n z_1^{m-n}p_n\left(z_1z_2;q^{m-n},b|q\right),$$

where $p_n(x;a,b|q)$ is the little Jacobi polynomials.

Theorem 9. *For* $0 < q < 1$ *and* $bq < 1$*, the polynomials* $\{p_{m,n}(z,\overline{z};b|q)\}$ *satisfy the following orthogonality*

$$\int_{\mathbb{C}} p_{m,n}(z,\overline{z};b|q)\,\overline{p_{s,t}(z,\overline{z};b|q)}\,d\mu(z,\overline{z}) = \frac{(bq;q)_\infty}{(q;q)_\infty}\frac{q^{mn}(q,bq;q)_m(q,bq;q)_n}{1-bq^{m+n+1}}\delta_{m,s}\delta_{n,t},$$

where

$$d\mu(z,\overline{z}) = \frac{d\theta}{2\pi}\otimes\sum_{k=0}^{\infty}\frac{(bq;q)_k\,q^k}{(q;q)_k}\delta\left(r-q^{k/2}\right),$$

$z = re^{i\theta}, r\in\mathbb{R}^+, \theta\in[0,2\pi]$ *and* $m,n,s,t\in\mathbb{N}_0$*.*

Raising operators, lowering operators, generating functions, $\cdots$.

Linearization of products: We are working on it.