Introduction to the a-number of graphs and hypergraphs

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Definition of “the” invariant

I and Dr. Hanchul Park [1] have introduced one graph invariant as follows:

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G : a finite undirected graph

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The signed a-number $sa(G)$ of G is defined recursively by:

$$sa(G) = \begin{cases} 1, & \text{if } G = \varnothing; \\ 0, & \text{if } G \text{ has a connected component} \\ & \quad \text{on odd vertices;} \\ -\sum_{I \subsetneq V} sa(G|_I), & \text{otherwise,} \end{cases}$$

where V is the vertex set of G.

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where V is the vertex set of G.

Remark. If $G_1, \ldots, G_k$ are the connected components of G, then

$$sa(G) = \prod_{i=1}^{k} sa(G_i).$$

Invariants

Definition

The a-,a_i-, and b- numbers of a graph G, denoted by $a(G)$, $a_i(G)$, $b(G)$, are defined by

$$a(G) = |sa(G)| = (-1)^{|V|/2} sa(G),$$

$$a_i(G) = \sum_{\substack{I \subseteq V \\ |I| = 2i}} a(G|_I),$$

$$b(G) = \sum_{I \subseteq V} sa(G|_I).$$

Remark. $a(G) = a_{|V|/2}(G)$

Examples: path graphs

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If $G =$ ,

H	$sa(H)$	count	
$\varnothing$	1	1	1
	-1	5	-5
	2	3	6
	1	3	3

the sum is 5 and $sa(G) = -5$.

Another description

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Then $\hat{S}_G$ is a graded poset.

Consider a Möbius function μ of $\hat{S}_G$:

$$\mu(x,y) = \begin{cases} 1, & \text{if } x = y; \\ -\sum_{x \le z < y} \mu(x,z), & \text{if } x < y. \end{cases}$$

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Then,

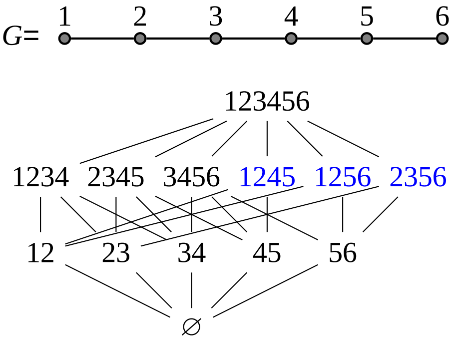
$$sa(G) = \mu(\varnothing, V)$$

and $a_i(G)$ is the i-th Whitney number of first kind of $\hat{S}_G$.

Example revisit

$G=$ 

Example revisit



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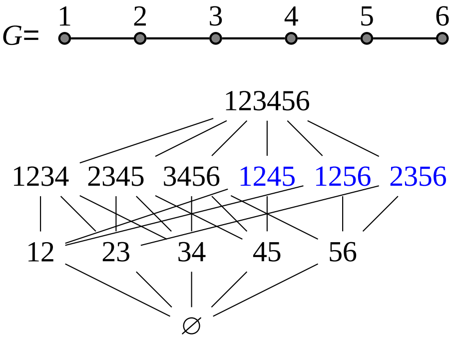

1

Furthermore, a_i-number of G is $(1, 5, 9, 5)$.

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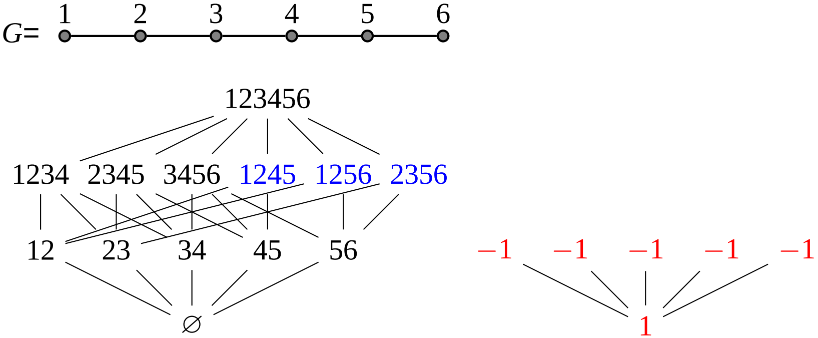

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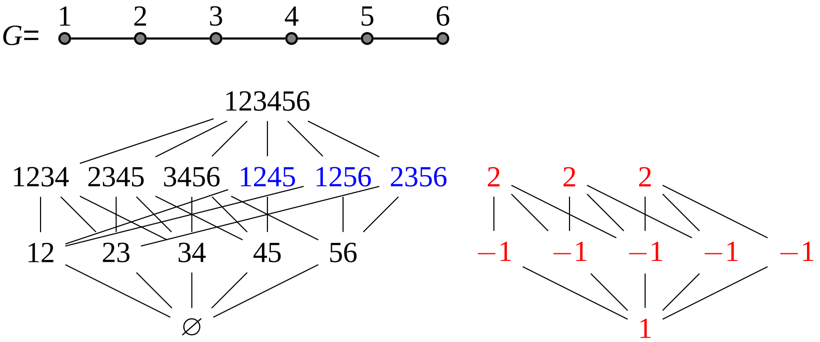

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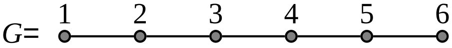

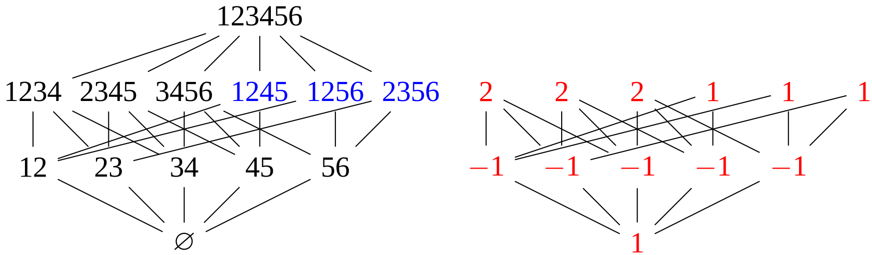

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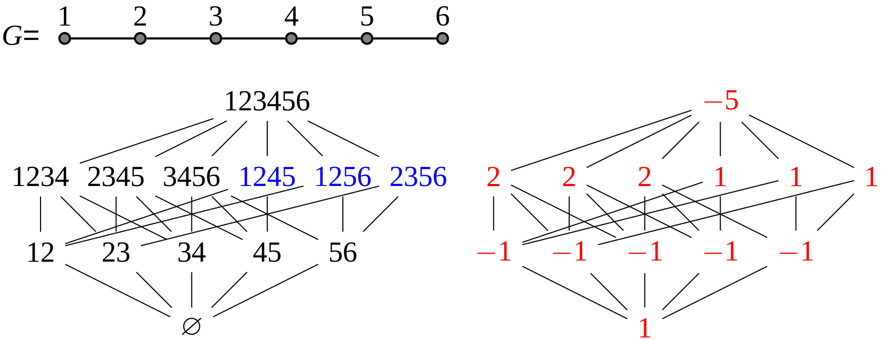

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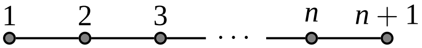
Path graph

$G =$ 

Proposition

If $G = P_{n+1}$ is a path graph, then

$$a_i(P_{n+1}) = \binom{n+1}{i} - \binom{n+1}{i-1}$$

and

$$b(P_{n+1}) = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ (-1)^{\frac{n}{2}} C_{\frac{n}{2}}, & \text{if } n \text{ is even,} \end{cases}$$

where $C_k = \frac{1}{k+1}\binom{2k}{k}$ is the k-th Catalan number.

<u>Remark</u>

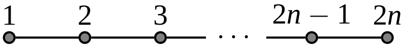
- $a(P_{2n})$ and $b(P_{2n+1})$ are new ways to obtain the Catalan numbers.

$a_i(P_n)$	$i=0$	1	2	3	4	5
$n=0$	1					
1	1					
2	1	1				
3	1	2				
4	1	3	2			
5	1	4	5			
6	1	5	9	5		
7	1	6	14	14		
8	1	7	20	28	14	
9	1	8	27	48	42	
10	1	9	35	75	90	42

Table : Values of $a_i(P_n)$ make up a Catalan triangle.

Proof

Let us show that $sa(G) = (-1)^n \mathcal{C}_n$ for $G = P_{2n}$.

$G =$ 

Pick two vertices of G, named p and q ($1 \leq p < q \leq 2n$).
Consider all $I \subseteq [2n]$ s.t. p and q are the first two vertices not in I:

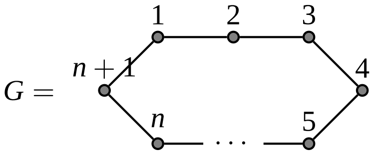
$$S(p,q) = \sum_I sa(G|_I) = sa(P_{p-1}) \cdot sa(P_{q-p-1}) \cdot b(P_{2n-q}).$$

Note that if either p is even or $q \neq 2n$, then $S(p,q) = 0$. Hence, by summing $S(p,q)$ whenever p is odd and $q = 2n$, we have

$$-sa(P_{2n}) = sa(P_0)sa(P_{2n-2}) + sa(P_2)sa(P_{2n-4}) + \cdots + sa(P_{2n-2})sa(P_0),$$

the famous recurrence relation for the (signed) Catalan number.

Cyclic graph



Proposition

If $G = C_{n+1}$ is a cycle graph, then

$$a_i(C_{n+1}) = \begin{cases} \binom{n+1}{i}, & \text{if } 2i < n+1; \\ \frac{1}{2}\binom{2i}{i}, & \text{if } 2i = n+1. \end{cases}$$

and

$$b(C_{n+1}) = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ (-1)^{\frac{n}{2}}\binom{n}{n/2}, & \text{if } n \text{ is even.} \end{cases}$$

$a_i(C_n)$	$i = 0$	1	2	3	4	5
$n = 0$	1					
1	1					
2	1	1				
3	1	3				
4	1	4	3			
5	1	5	10			
6	1	6	15	10		
7	1	7	21	35		
8	1	8	28	56	35	
9	1	9	36	84	126	
10	1	10	45	120	210	126

Table : Values of $a_i(C_n)$ make up a half of the Pascal triangle.

Complete graph

- A_n: Euler zigzag number, i.e.,

$$\sec x + \tan x = \sum_{n=0}^{\infty} A_n \frac{x^n}{n!}.$$

- A_{2i} are secant numbers and A_{2i+1} are tangent numbers.

Proposition

If $G = K_{n+1}$ is a complete graph, then

$$a_i(K_{n+1}) = \binom{n+1}{2i} A_{2i}$$

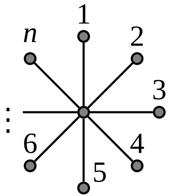
and

$$b(K_{n+1}) = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ (-1)^{\frac{n}{2}} A_{n+1}, & \text{if } n \text{ is even,} \end{cases}$$

$a_i(K_n)$	$i=0$	1	2	3	4
$n=0$	1				
1	1				
2	1	1			
3	1	3			
4	1	6	5		
5	1	10	25		
6	1	15	75	61	
7	1	21	175	427	
8	1	28	350	1708	1385

Table : Values of $a_i(K_n)$ make up coefficients of the Swiss-Knife polynomials.

Star graph

$G =$ 

Proposition

If $G = K_{1,n}$ is a star graph, then

$$a_i(K_{1,n}) = \begin{cases} \binom{n}{2i-1} A_{2i-1}, & \text{if } i \neq 0; \\ 1, & \text{if } i = 0. \end{cases}$$

and

$$b(K_{1,n}) = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ (-1)^{\frac{n}{2}} A_n, & \text{if } n \text{ is even,} \end{cases}$$

$a_i(K_{1,n-1})$	$i=0$	1	2	3	4
$n=0$	1				
1	1				
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4	1	3	2		
5	1	4	8		
6	1	5	20	16	
7	1	6	40	96	
8	1	7	70	336	272

Table : Values of $a_i(K_{1,n-1})$.

Complete multipartite graphs

$K_{p_1,\ldots,p_m}$: the complete m-partite graph with p_1-set, ..., p_m-set. [2]

Theorem (Seo-Shin (2012))

$$\sum_{p_1,\ldots,p_m\geq 0} sa(K_{p_1,\ldots,p_m})\frac{x_1^{p_1}}{p_1!}\cdots\frac{x_m^{p_m}}{p_m!} = \left(\frac{(1-m)+\cosh x_1+\cdots+\cosh x_m}{\cosh(x_1+\cdots+x_m)}\right)$$

- $\sum_{n\geq 0} sa(K_n)\frac{x^n}{n!} = \operatorname{sech} x$
- $\sum_{n\geq 0} sa(K_{1,n};t)\frac{x^n}{n!} = -\tanh x$

[2] S. Seo, H. Shin, Signed a-polynomials of graphs and Poincaré polynomials of real toric manifolds, arXiv:1212.6307

- union $G \amalg v$: add a single isolated vertex v to G
- join $G + v$: add a single dominating vertex v to G, i.e., v is connected to all other vertices

Proposition (C-Kim (on progress))

$a_i(G \amalg v) = a_i(G)$ *and*

$$a_i(G+v) = a_i(G) + \sum_{k=0}^{i-1} \binom{n-2k}{2i-2k-1} A_{2i-2k-1} a_k(G)$$

Since $a_i(G+v)$ is a linear combination of $a_k(G)$, we can compute the a_i-number of a threshold graph obtainable from a vertex by repeated applications of joins and unions.

Remark Put $G = K_n$, then we obtain

$$A_{2i} = \sum_{k=0}^{i-1} \binom{2i-1}{2k} A_{2i-2k-1} A_{2k}.$$

Motivation

- A toric variety of $\dim_{\mathbb{C}} n$: a normal algebraic variety over $\mathbb{C}$ with effective $(\mathbb{C}^*)^n$-action having an open dense orbit ($\mathbb{C}^* = \mathbb{C} \setminus \{0\}$).
- The real points of a projective toric variety is a submanifold called a real toric manifold.

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- The real points of a projective toric variety is a submanifold called a real toric manifold.
- A simple polytope $P^n \subset \mathbb{R}^n$ is Delzant if outward normal vectors of facets form an integral basis at each vertex.

Fact

A Delzant polytope determines a non-singular compact toric variety.

Preliminaries: Building sets

We introduce one interesting class of Delzant polytopes.

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Definition

A building set $\mathcal{B}$ on $[n+1] := \{1, \ldots, n+1\}$:
$\mathcal{B}$: a set of nonempty subsets of $[n+1]$ such that

1. $\{1\}, \ldots, \{n+1\} \in \mathcal{B}$,
2. if $I, J \in \mathcal{B}$ and $I \cap J \neq \varnothing$, then $I \cup J \in \mathcal{B}$.

$\mathcal{B}$ is connected if $[n+1] \in \mathcal{B}$.

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Example

$\mathcal{B} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{2,3,4\}, \{1,2,3,4\}\}$
$= \{1, 2, 3, 4, 12, 234, 1234\}$: connected building set on $[4]$.
$\mathcal{B}' = \{1, 2, 3, 4, 12, 34\}$: building set, not connected

Preliminaries: Graphical building sets

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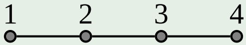
$\mathcal{B}(G)$ is indeed a building set.
$\mathcal{B}(G)$ is connected $\Leftrightarrow$ G is connected.

Example

$G =$ 

$\mathcal{B}(G) = \{1, 2, 3, 4, 12, 23, 34, 123, 234, 1234\}$
13 or 124 are not elements of $\mathcal{B}(G)$.

Nestohedra and graph associahedra

- $A, B \subset \mathbb{R}^n$, the Minkowski sum $A + B$ is

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The nestohedron $P_\mathcal{B}$: the Minkowski sum

$$P_\mathcal{B} = \sum_{I \in \mathcal{B}} \Delta_I \subset \mathbb{R}^{n+1}.$$

If $\mathcal{B} = \mathcal{B}(G)$ is graphical, then $P(G) := P_\mathcal{B}$ is a graph associahedron.

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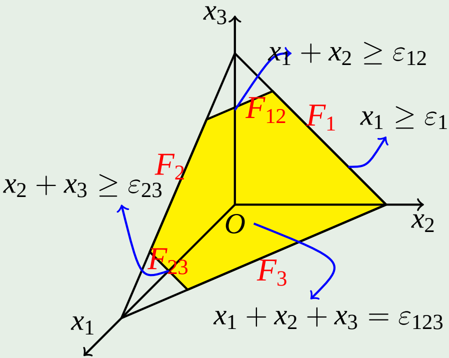
Theorem (Zelevinsky 2006)

Every nestohedron is a Delzant polytope. Indeed, there is natural way to make $P_{\mathcal{B}}$ be Delzant by cutting off from simplex.

Example

$\mathcal{B} = \{1, 2, 3, 12, 23, 123\}$. $P_{\mathcal{B}}$ is a pentagon.
$\forall$ element of $\mathcal{B}$ other than 123 indicates a facet.



Example

Associahedron As^3 (or Stasheff polytope)

$G =$ 

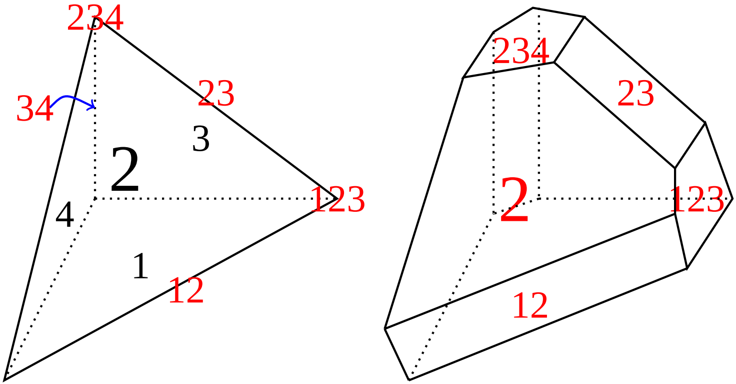
$\mathcal{B} = \{1, 2, 3, 4, 12, 23, 34, 123, 234, 1234\}$

Example

Associahedron As^3 (or Stasheff polytope)

$G =$ 1 — 2 — 3 — 4 — 5 — 6

$\mathcal{B} = \{1, 2, 3, 4, 12, 23, 34, 123, 234, 1234\}$



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$M(G)$: real toric variety rising from a graph G

Remark Toric varieties corr. to K_n is known as Hessenberg varieties, and $M(K_n)$ is called a real Hessenberg variety.

Motivation

Theorem (Henderson 2010, Suciu 2012)

$$\beta_i(M(K_{n+1})) = A_{2i}\binom{n+1}{2i} (= a_i(K_{n+1})),$$

where β_i is the i-th Betti number and A_{2i} is the Euler secant number 1, 5, 61, 1385, $\ldots$

n	β_0	β_1	β_2	β_3	β_4	β_5	β_6	β_7
1	1	1						
2	1	3	0					
3	1	6	5	0				
4	1	10	25	0	0			
5	1	15	75	61	0	0		
6	1	21	175	427	0	0	0	
7	1	28	350	1708	1385	0	0	0

Table : Betti numbers of $M(K_{n+1})$.

Main result

- G : a simple graph
- $\beta_i(X)$: the i-th rational Betti number of X
- $\chi(X)$: the Euler characteristic of X

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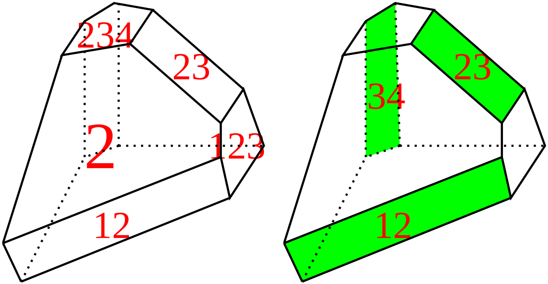
Theorem (C-Park (2012))

$$\beta_i(M(G)) = a_i(G) \quad \textit{and} \quad \chi(M(G)) = b(G)$$

Furthermore, $H_*(M(G);\mathbb{Z})$ has no odd-torsion.

Sketch of proof

We recall every facet of $P(G)$ is indexed by sets of vertices of G. We define $P_G^{odd} = \bigcup_{|I|=odd} F_I$ and $P_G^{even} = \bigcup_{|I|=even} F_I$.



Step 1. $\beta_i(M(G)) = \sum_{\substack{H \subseteq G \\ |H|=even}} \beta_{i-1}(P_H^{odd})$

- The formula of Suciu-Trevisan for rational homology of real toric varieties
- + tricky.

Step 2. Observe that

$$\begin{array}{ccc} P_H^{odd} & \xleftrightarrow{\text{Alexander Duality}} & P_H^{even} \\ {\scriptstyle\text{dual complex}}\updownarrow & & \updownarrow \\ K_H^{odd} & \longleftrightarrow & K_H^{even} \end{array} .$$

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\end{array}
.
$$

Step 3.

1. $\hat{S}_H = \{I \subset V \mid H|_I$ has no odd connected component$\}$ as a poset ordered by inclusions (V : the vertex set of H, $|V| = 2k$)
2. $S_H = \hat{S}_H \setminus \{\varnothing, V\}$
3. L_H^{even} : order complex of S_H

Then, L_H^{even} is a subdivision of K_H^{even}. This implies that $|K_H^{even}| = |L_H^{even}|$.

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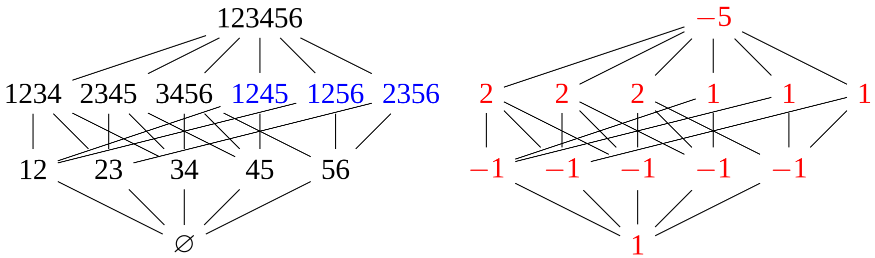
3. L_H^{even} : order complex of S_H

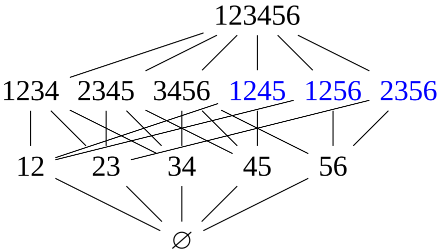
Then, L_H^{even} is a subdivision of K_H^{even}. This implies that $|K_H^{even}| = |L_H^{even}|$.

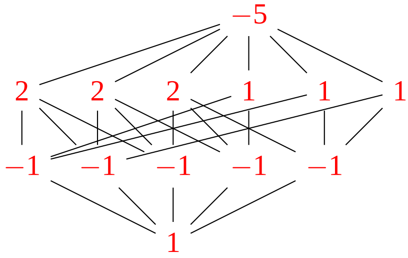
Step 4. $\hat{S}_H$ is shellable. Hence, $|L_H^{even}| \simeq \bigvee^a S^{k-2}$. Moreover, $a = |\mu(\varnothing, V)| = a(H)$.

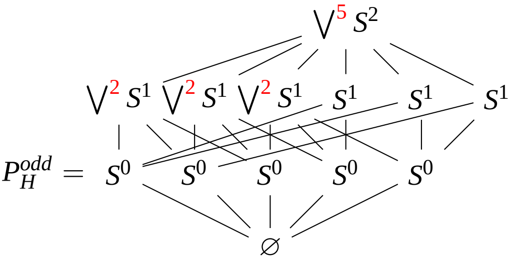
Example


123456
1234
2345
3456
1245
1256
2356
12
23
34
45
56
∅
−5
2
2
2
1
1
1
−1
−1
−1
−1
−1
1

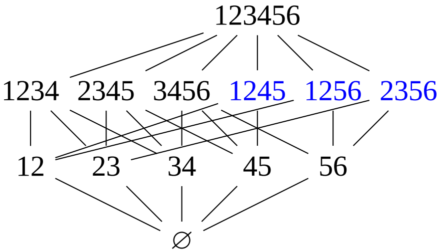
Example

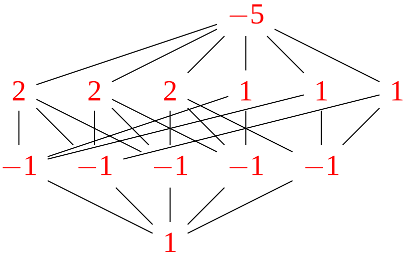

123456
1234 2345 3456 1245 1256 2356
12 23 34 45 56
∅

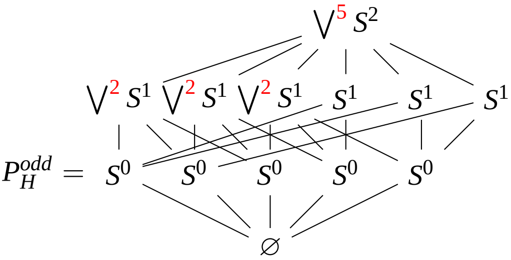

−5
2 2 2 1 1 1
−1 −1 −1 −1 −1
1


$P_H^{odd} =$
$\bigvee^5 S^2$
$\bigvee^2 S^1$ $\bigvee^2 S^1$ $\bigvee^2 S^1$ S^1 S^1 S^1
S^0 S^0 S^0 S^0 S^0
∅

Example







$$P^{odd}_{123456} \simeq S^4 \setminus P^{even}_{123456} = S^4 \setminus |L^{even}_{123456}|$$

$$\simeq S^4 \setminus \left(\bigvee^5 S^1\right) \simeq \bigvee^5 S^2 \text{ (Alexander duality)}$$

$$\beta_i(M(G)) = \sum_{\substack{H\subseteq G \\ |H|=even}} \beta_{i-1}(P_H^{odd})$$

Nestohedra

Question

Can we define such a-number for nestohedra?

Building set	$\longleftrightarrow$	Graphical building set
Nestohedron	$\longleftrightarrow$	Graph associahedron
Hypergraph	$\longleftrightarrow$	Graph

Hypergraph $G = (V, E)$: E is a collection of subsets of V.
If E is a collection of 2-subsets of V, then G is an ordinary graph.

Example

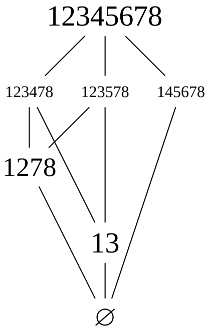
$G = ([8], \{13, 1278, 123478, 123578, 145678\})$

Building set $\mathcal{B} = \left\{ \begin{array}{c} 1, 2, 3, 4, 5, 6, 7, 8, 13, 1278, 12378, 123478, 123578 \\ 145678, 1234578, 1345678, 1245678, 12345678 \end{array} \right\}$

Example

$$\mathcal{B} = \left\{ \begin{array}{c} 1, 2, 3, 4, 5, 6, 7, 8, 13, 1278, 12378, 123478, 123578 \\ 145678, 1234578, 1345678, 1245678, 12345678 \end{array} \right\}$$

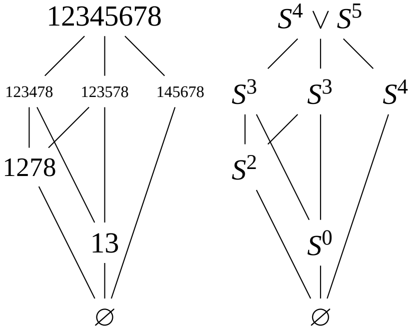
$$\hat{S}_{\mathcal{B}} = \{13, 1278, 123478, 123578, 145678, 12345678\}$$



Example

$$\mathcal{B} = \left\{ \begin{array}{c} 1, 2, 3, 4, 5, 6, 7, 8, 13, 1278, 12378, 123478, 123578 \\ 145678, 1234578, 1345678, 1245678, 12345678 \end{array} \right\}$$

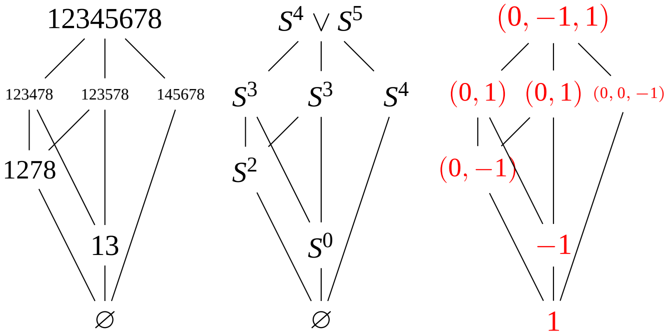
$$\hat{S}_{\mathcal{B}} = \{13, 1278, 123478, 123578, 145678, 12345678\}$$



Example

$$\mathcal{B} = \left\{ \begin{array}{c} 1, 2, 3, 4, 5, 6, 7, 8, 13, 1278, 12378, 123478, 123578 \\ 145678, 1234578, 1345678, 1245678, 12345678 \end{array} \right\}$$

$$\hat{S}_{\mathcal{B}} = \{13, 1278, 123478, 123578, 145678, 12345678\}$$



Work in progress with Hanchul Park and Kyoung Sook Park:

Remark $\hat{S}_{\mathcal{B}}$ is not shellable and pure in general.

Question

For a nestohedron, is the homotopy type of P_H^{odd} wedge of spheres?

If "YES", we can define a-vector as above : each component is the number of spheres of each dim consisting the homotopy type of P_H^{odd}.

Question

How to compute a-vector?