# Littlewood-Richardson numbers of Schur's $\mathsf{S}$- and $\mathsf{P}$-functions

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S. Cho and D. Moon, *Reduction formulae of Littlewood-Richardson coefficients*, Adv. in Appl. Math. 46 (2011) 125 – 143

S. Cho, *A new Littlewood-Richardson rule for Schur* $\mathsf{P}$*-functions*, Trans. Amer. Math. Soc. 365 (2013) 939 – 972

Partition

$\lambda = (\lambda_1, \ldots, \lambda_l) \vdash k$ :

$$\lambda_1 \geq \cdots \geq \lambda_l \geq 0, \ \sum \lambda_i = k$$

Schur function [Jacobi 1841, Schur 1901]

$$s_\lambda(x_1, x_2, \ldots, x_n) = \frac{\left| x_i^{\lambda_j + n - j} \right|}{\left| x_i^{n-j} \right|}$$

$s_\lambda(x_1, x_2, \ldots, x_n)$ is a symmetric polynomial:

$$s_\lambda(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)}) = s_\lambda(x_1, x_2, \ldots, x_n) \text{ for all } \pi \in \mathfrak{S}_n$$

# Ring of Symmetric functions

$$\Lambda = \mathbb{C}[m_\lambda] = \bigoplus_{n \geq 0} \Lambda^n$$

$m_\lambda(x) = \sum x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_\ell}^{\lambda_\ell}$ is a monomial symmetric function

- powersum symmetric functions
  $p_{(3,2)} = (x_1^3 + x_2^3 + \cdots)(x_1^2 + x_2^2 + \cdots)$
- elementary symmetric functions
  $e_{(3,2)} = (x_1x_2x_3 + x_1x_2x_4 + \cdots)(x_1x_2 + x_1x_3 + x_2x_3 \cdots)$
- complete homogeneous symmetric functions
  $h_{(3,2)} = (x_1^3 + \cdots + x_1^2x_2 + \cdots + x_1x_2x_3 + \cdots)(x_1^2 + \cdots + x_1x_2 + \cdots)$
- Schur functions

# Combinatorial model

Semistandard tableau of shape $\lambda = (4, 3, 3)$:

| | | | |
|---|---|---|---|
| 1 | 2 | 2 | 3 |
| 2 | 5 | 6 | |
| 4 | 6 | 8 | |

$$s_\lambda = \sum_T x^T, \text{ T: semistandard tableau}$$

$$s_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + 2x_1 x_2 x_3 + \cdots$$

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 2 | 1 | 1 | 1 | 3 | 1 | 2 | 1 | 3 |
| 2 | | 2 | | 3 | | 3 | | 3 | | 2 | |

# Jacobi-Trudi determinants

$$s_\lambda = |h_{\lambda_i - i + j}| = |s_{\lambda_i - i + j}|$$

$$s_{(2,1)} = \begin{vmatrix} s_2 & s_3 \\ s_0 & s_1 \end{vmatrix} = s_2 s_1 - s_3 s_0$$

$$s_{\tilde{\lambda}} = |e_{\lambda_i - i + j}| = |s_{1^{\lambda_i - i + j}}|$$

$$s_{(2,1)} = s_{\widetilde{(2,1)}} = \begin{vmatrix} s_{(1,1)} & s_{(1,1,1)} \\ s_0 & s_1 \end{vmatrix} = s_{(1,1)} s_1 - s_{(1,1,1)} s_0$$

## (isomorphic) graded algebras

- $\Lambda = \bigoplus_n \Lambda^n$: ring of symmetric functions

$$s_\lambda(x)s_\mu(x) = \sum_\nu c_{\lambda,\mu}^\nu s_\nu(x)$$

- $R = \bigoplus_n R^n$: irreducible representations of $\mathfrak{S}_n$'s

$$\chi^\lambda \cdot \chi^\mu = \sum_\nu c_{\lambda,\mu}^\nu \chi^\nu$$

- finite dimensional polynomial representations of $GL_m(\mathbb{C})$

$$V(\lambda) \otimes V(\mu) = \bigoplus_\nu c_{\lambda,\mu}^\nu V(\nu)$$

- $H^*(Gr(m,n))$: cohomology ring of a Grassmannian

$$\sigma_\lambda \sigma_\mu = \sum_\nu c_{\lambda,\mu}^\nu \sigma_\nu$$

# Littlewood-Richardson numbers (1)

LR-tableaux [Littlewood & Richardson 1934]

Semi-standard tableaux of shape $\nu/\lambda$ with content $\mu$, whose row word is a lattice word

$\nu = (7, 6, 5, 4), \lambda = (4, 2, 2, 0),\ \mu = (5, 5, 2, 2)$

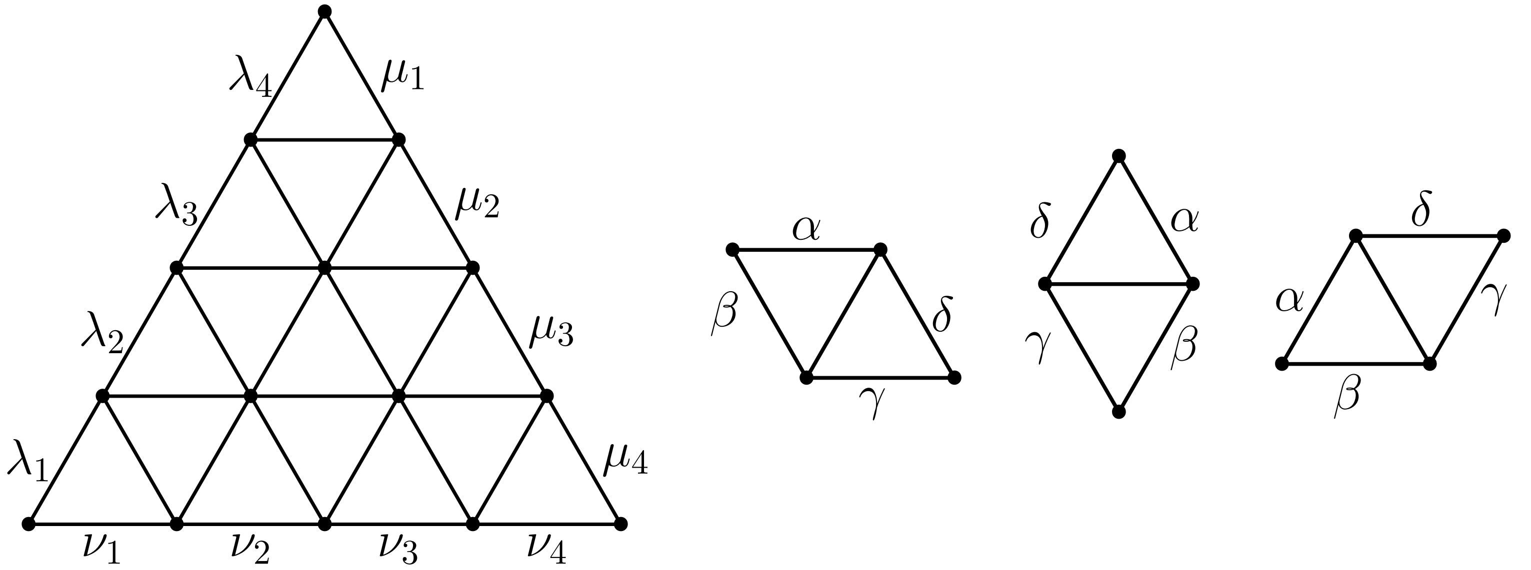
T =

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | 1 | 1 | 1 |
| | | 1 | 1 | 2 | 2 | |
| | | 2 | 3 | 3 | | |
| 2 | 2 | 4 | 4 | | | |

$c^{\nu}_{\lambda,\mu} =$ number of LR-tableaux

# Littlewood-Richardson numbers (2)

Hives [Knutson & Tao 1999]

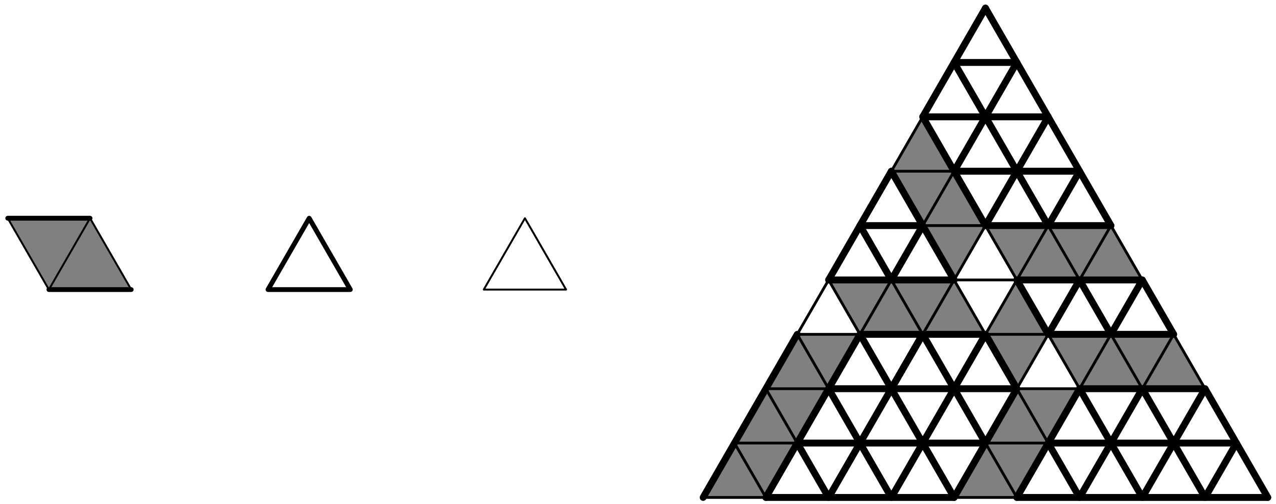



$\alpha \geq \gamma$, $\beta \geq \delta$, and in each triangle the sum of two values on oblique sides is same as the value on the horizontal side.

$c^{\nu}_{\lambda,\mu} =$ number of hives

# Littlewood-Richardson numbers (3)

Puzzles [Knutson, Tao & Woodward 2004]



$\lambda = (4,2), \mu = (3,2), \nu = (7,4) \subseteq (7,7)$

$c^{\nu}_{\lambda,\mu} =$ number of puzzles

# Littlewood-Richardson numbers (4)

[Stembridge 2002]

$c^{\nu}_{\lambda,\mu}$ = number of semistandard tableaux of shape $\mu$ such that

$\lambda + wt(T) = \nu$ and $\lambda + wt(T_{\geq j})$ is a partition for all $j \geq 1$

[Vakil 2006] Checker games

[Coskun 2009] Mondrian tableaux

# Important Theorems

**Horn's inequalities** [Klyachko 1998, Knutson & Tao 1999] Let $\lambda, \mu$ and $\nu$ be partitions of lengths at most $n$. Then $c^{\nu}_{\lambda\mu} > 0$ if and only if $|\nu| = |\lambda| + |\mu|$ and for all $r \leq n$, $\sum_{k \in K} \nu_k \leq \sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j$ holds for all $(I, J, K) \in R^n_r$.

**Saturation** [Knutson & Tao 1999] For a positive integer $N$,

$$c^{\nu}_{\lambda\mu} > 0 \quad \text{if and only if} \quad c^{N\nu}_{N\lambda N\mu} > 0$$

**Fulton's conjecture** [Knutson, Tao & Woodwards 2004] For a positive integer $N$,

$$c^{\nu}_{\lambda\mu} = 1 \quad \text{if and only if} \quad c^{N\nu}_{N\lambda N\mu} = 1$$

**Interior** [Knutson, Tao, Woodwards 2004] If $\lambda, \mu, \nu$ are partitions with $n$ distinct parts, and each of the Horn inequalities holds strictly, then $c^{\nu}_{\lambda\mu}$ is at least 2.

# Identities of LR-numbers

**Symmetry** $c^{\nu}_{\lambda,\mu} = c^{\nu}_{\mu,\lambda}$

**Conjugation** $c^{\nu}_{\lambda,\mu} = c^{\widetilde{\nu}}_{\widetilde{\lambda},\widetilde{\mu}}$

**Reduction I** For any three indices $0 \le i, j, k \le n$ with $i + j = k + n$, if $\lambda_i + \mu_j = \nu_k$ then $c^{\nu}_{\lambda\mu} = c^{\nu-\nu_k}_{\lambda-\lambda_i,\mu-\mu_j}$.

**Reduction II** If there are $\lambda_i, \mu_j, \nu_k$ with $i + j = k - 1$ such that $\lambda_{i+1} < \lambda_i$, $\mu_{j+1} < \mu_j$, $\nu_k < \nu_{k-1}$ and $\lambda_i + \mu_j \ge \nu_1 + \nu_k + 1$, then $c^{\nu}_{\lambda,\mu} = c^{\nu-(1^{k-1})}_{\lambda-(1^i),\ \mu-(1^j)}$.

**Factorization** If $c^{\nu}_{\lambda\mu} > 0$ and there exists $(I, J, K) \in R^n_r$ for some $r < n$ such that $\sum_{k\in K} \nu_k = \sum_{i\in I} \lambda_i + \sum_{j\in J} \mu_j$, then $c^{\nu}_{\lambda\mu} = c^{\nu_K}_{\lambda_I \mu_J} c^{\nu_{K^c}}_{\lambda_{I^c} \mu_{J^c}}$ .

# Combinatorial proofs

**Symmetry** Benkart, Sottile and Stroomer, "Tableau switching: algorithms and applications," JCTA 1996

**Conjugation** Hanlon and Sundaram, "On a bijection between Littlewood-Richardson fillings of conjugate shape," JCTA 1992

**Reductions** C, Jung and Moon,

"A combinatorial proof of the reduction formula for Littlewood-Richardson coefficients," JCTA 2007

"A bijective proof of the second reduction formula for Littlewood-Richardson coefficients," BKMS 2008

"A hive-model proof of the second reduction formula of Littlewood-Richardson coefficients," Ann. Comb. 2011

**Factorization** King, Tollu and Toumazet "Factorisation of Littlewood-Richardson coefficients," JCTA 2009

# Reductions: examples

**Reduction I** When $\lambda = (5, 4, 4, 3, 3, 2, 1, 0)$,

$\mu = (5, 5, 5, 4, 4, 1, 0, 0)$, $\nu = (8, 8, 6, 6, 5, 5, 4, 4)$

and $i = 6$, $j = 5$, $k = 3$,

$$i + j = k + n \quad \text{and} \quad \lambda_6 + \mu_5 = \nu_3 . \quad \text{Hence}$$

$$c^{(8,8,6,6,5,5,4,4)}_{(5,4,4,3,3,2,1,0),(5,5,5,4,4,1,0,0)} = c^{(8,8,6,5,5,4,4)}_{(5,4,4,3,3,1,0),(5,5,5,4,1,0,0)}$$

**Reduction II** When $\lambda = (7, 6, 5, 3, 1, 0, 0, 0)$,

$\mu = (6, 5, 5, 5, 3, 0, 0, 0)$, $\nu = (8, 8, 8, 8, 6, 4, 2, 2)$ and $i = 2$, $j = 4$,

$k = 7$, we have $\lambda_2 > \lambda_3$, $\mu_4 > \mu_5$, $\nu_6 > \nu_7$ and

$$i + j = k - 1, \quad \lambda_2 + \mu_4 \geq \nu_1 + \nu_7 + 1. \quad \text{Hence}$$

$$c^{(8,8,8,8,6,4,2,2)}_{(7,6,5,3,1,0,0,0),(6,5,5,5,3,0,0,0)} = c^{(7,7,7,7,5,3,2,2)}_{(6,5,5,3,1,0,0,0),(5,4,4,4,3,0,0,0)}$$

## Reduction from factorization: Observations

- If $\lambda$, $\mu$, $\nu$ are partitions of length at most 2, then $c^{\nu}_{\lambda,\mu} = 1$ if it is nonzero.

- It is not easy to check the condition $(I, J, K) \in R^n_r$. However, we can explicitly write down the conditions for $(I, J, K)$ to be in $R^n_r$ when $r = 1, 2, n-1$ or $n-2$.

- Factorization is a reduction if $r = 1, 2, n-1$ or $n-2$.

## **Condition** $(I, J, K) \in R_r^n$

For $I = \{i_1 < i_2 < \cdots < i_r\}$,

$$\pi(I) := (i_r - r, i_{r-1} - (r-1), \ldots, i_1 - 1)$$

For I, J, K: $r$-subsets of $[n]$,

$$(I, J, K) \in R_r^n \quad \text{if and only if} \quad c_{\pi(I),\pi(J)}^{\pi(K)} = 1$$

## Condition $(I, J, K) \in R^n_r$ when $r = 1, 2$

(1) $(\{i\}, \{j\}, \{k\}) \in R^n_1$ if and only if $i + j = k + 1$.

(2) $(\{i_1, i_2\}, \{j_1, j_2\}, \{k_1, k_2\}) \in R^n_2$ if and only if the following conditions are satisfied:

$$
\begin{aligned}
i_1 + i_2 + j_1 + j_2 &= k_1 + k_2 + 3\,, \\
i_1 + j_1 &\leq k_1 + 1\,, \\
i_1 + j_2 &\leq k_2 + 1\,, \\
i_2 + j_1 &\leq k_2 + 1\,.
\end{aligned}
$$

## **Condition** $(I, J, K) \in R^n_r$ **when** $r = n-1, n-2$

(1) $(\{i\}^c, \{j\}^c, \{k\}^c) \in R^n_{n-1}$ if and only if $i + j = k + n$.

factorization = reduction formula I !!

(2) $(\{i_1, i_2\}^c, \{j_1, j_2\}^c, \{k_1, k_2\}^c) \in R^n_{n-2}$ if and only if the following conditions are satisfied:

$$
\begin{aligned}
i_1 + i_2 + j_1 + j_2 &= k_1 + k_2 + 2n - 1\,, \\
k_2 + n &\leq i_2 + j_2\,, \\
k_1 + n &\leq i_2 + j_1\,, \\
k_1 + n &\leq i_1 + j_2\,.
\end{aligned}
$$

## (Extended) Reduction formulae [C, Moon 2011]

$(\mathbf{r=1})$ Suppose that there are $1 \leq i, j, k \leq n$ satisfying the following condition;

$$(*) \quad i + j \leq k + 1\,.$$

a) If $\lambda_i + \mu_j < \nu_k$, then $c_{\lambda,\mu}^{\nu} = 0$.

b) If $\lambda_i + \mu_j = \nu_k$, then $c_{\lambda,\mu}^{\nu} = c_{\lambda-\lambda_{\{i\}},\ \mu-\mu_{\{j\}}}^{\nu-\nu_{\{k\}}}$.

$(\mathbf{r=n-1})$ Suppose there are $1 \leq i, j, k \leq n$ satisfying the following condition;

$$(*) \quad i + j \geq k + n\,.$$

a) If $\lambda_i + \mu_j > \nu_k$, then $c_{\lambda,\mu}^{\nu} = 0$.

b) If $\lambda_i + \mu_j = \nu_k$, then $c_{\lambda,\mu}^{\nu} = c_{\lambda-\lambda_{\{i\}},\ \mu-\mu_{\{j\}}}^{\nu-\nu_{\{k\}}}$.

**(r = 2)** Suppose that there are $1 \leq i_1, i_2, j_1, j_2, k_1, k_2 \leq n$ such that $i_1 < i_2$, $j_1 < j_2$, $k_1 < k_2$ satisfying the following conditions;

$$
(*) \quad \begin{aligned}
i_1 + i_2 + j_1 + j_2 &\leq k_1 + k_2 + 3\,, \\
i_1 + j_1 &\leq k_1 + 1\,, \\
i_1 + j_2 &\leq k_2 + 1\,, \\
i_2 + j_1 &\leq k_2 + 1\,.
\end{aligned}
$$

a) If $\lambda_{i_1} + \lambda_{i_2} + \mu_{j_1} + \mu_{j_2} < \nu_{k_1} + \nu_{k_2}$, then $c^{\nu}_{\lambda,\mu} = 0$.

b) If $\lambda_{i_1} + \lambda_{i_2} + \mu_{j_1} + \mu_{j_2} = \nu_{k_1} + \nu_{k_2}$, then

$$
c^{\nu}_{\lambda,\mu} = \begin{cases}
c^{\nu - \nu_{\{k_1,k_2\}}}_{\lambda - \lambda_{\{i_1,i_2\}},\ \mu - \mu_{\{j_1,j_2\}}} & \text{if } \ \nu_{k_2} - \lambda_{i_1} \leq \mu_{j_2} \\
 & \mu_{j_2} \leq \min\{\nu_{k_2} - \lambda_{i_2}, \nu_{k_1} - \lambda_{i_1}\} \\
 & \quad \text{and } \ \nu_{k_1} - \lambda_{i_1} \leq \mu_{j_1} \leq \nu_{k_1} - \lambda_{i_2}, \\
0 & \text{otherwise.}
\end{cases}
$$

$(\mathbf{r = n - 2})$ Suppose that there are $1 \leq i_1, i_2, j_1, j_2, k_1, k_2 \leq n$ such that $i_1 < i_2$, $j_1 < j_2$, $k_1 < k_2$ satisfying the following conditions;

$$(*) \quad \begin{aligned} i_1 + i_2 + j_1 + j_2 &\geq k_1 + k_2 + 2n - 1\,, \\ k_2 + n &\leq i_2 + j_2\,, \\ k_1 + n &\leq i_2 + j_1\,, \\ k_1 + n &\leq i_1 + j_2\,. \end{aligned}$$

a) If $\lambda_{i_1} + \lambda_{i_2} + \mu_{j_1} + \mu_{j_2} > \nu_{k_1} + \nu_{k_2}$, then $c^{\nu}_{\lambda,\mu} = 0$.

b) If $\lambda_{i_1} + \lambda_{i_2} + \mu_{j_1} + \mu_{j_2} = \nu_{k_1} + \nu_{k_2}$, then

$$c^{\nu}_{\lambda,\mu} = \begin{cases} c^{\nu - \nu_{\{k_1,k_2\}}}_{\lambda - \lambda_{\{i_1,i_2\}},\ \mu - \mu_{\{j_1,j_2\}}} & \text{if } \ \nu_{k_2} - \lambda_{i_1} \leq \mu_{j_2} \\ & \mu_{j_2} \leq \min\{\nu_{k_2} - \lambda_{i_2}, \nu_{k_1} - \lambda_{i_1}\} \\ & \quad \text{and } \ \nu_{k_1} - \lambda_{i_1} \leq \mu_{j_1} \leq \nu_{k_1} - \lambda_{i_2}, \\ 0 & \text{otherwise.} \end{cases}$$

## Conjugated Reduction formulae [C, Moon 2011]

(**r = 1**) Suppose that there are $\lambda_i, \mu_j, \nu_k$ such that $\lambda_{i+1} < \lambda_i$, $\mu_{j+1} < \mu_j$ and $\nu_k < \nu_{k-1}$ satisfying the following condition;

$$(*) \quad \lambda_i + \mu_j \le \nu_k + 2\,.$$

a) If $i + j < k - 1$, then $c^{\nu}_{\lambda,\mu} = 0$.

b) If $i + j = k - 1$, then $c^{\nu}_{\lambda,\mu} = c^{\nu-(1^{k-1})}_{\lambda-(1^i),\ \mu-(1^j)}$.

(**r = n − 1**) Suppose that there are $\lambda_i, \mu_j, \nu_k$ such that $\lambda_{i+1} < \lambda_i$, $\mu_{j+1} < \mu_j$ and $\nu_k < \nu_{k-1}$ satisfying the following condition;

$$(*) \quad \lambda_i + \mu_j \ge \nu_1 + \nu_k + 1\,.$$

a) If $i + j > k - 1$, then $c^{\nu}_{\lambda,\mu} = 0$.

b) If $i + j = k - 1$, then $c^{\nu}_{\lambda,\mu} = c^{\nu-(1^{k-1})}_{\lambda-(1^i),\ \mu-(1^j)}$.

$(\mathbf{r = 2})$ Suppose that there are $\lambda_{i_1} > \lambda_{i_2}$, $\mu_{j_1} > \mu_{j_2}$, $\nu_{k_1} > \nu_{k_2}$ such that $\lambda_{i_1+1} < \lambda_{i_1}$, $\lambda_{i_2+1} < \lambda_{i_2}$, $\mu_{j_1+1} < \mu_{j_1}$, $\mu_{j_2+1} < \mu_{j_2}$ and $\nu_{k_1} < \nu_{k_1-1}$, $\nu_{k_2} < \nu_{k_2-1}$ satisfying the following conditions;

$$
\begin{aligned}
(*) \quad \lambda_{i_1} + \lambda_{i_2} + \mu_{j_1} + \mu_{j_2} &\leq \nu_{k_1} + \nu_{k_2} + 5\,, \\
\lambda_{i_2} + \mu_{j_2} &\leq \nu_{k_2} + 2\,, \\
\lambda_{i_2} + \mu_{j_1} &\leq \nu_{k_1} + 2\,, \\
\lambda_{i_1} + \mu_{j_2} &\leq \nu_{k_1} + 2\,.
\end{aligned}
$$

a) If $i_1 + i_2 + j_1 + j_2 < k_1 + k_2 - 2$, then $c^{\nu}_{\lambda,\mu} = 0$.

b) If $i_1 + i_2 + j_1 + j_2 = k_1 + k_2 - 2$, then

$$
c^{\nu}_{\lambda,\mu} = \begin{cases} c^{\nu-(2^{k_1-1},1^{k_2-k_1})}_{\lambda-(2^{i_1},1^{i_2-i_1}),\,\mu-(2^{j_1},1^{j_2-j_1})} & \text{if } \; k_2 - i_2 - 1 \leq j_2 \; \leq k_2 - i_1 - 1, \\ & k_1 - i_2 - 1 \leq j_1 \\ & j_1 \leq \min\{k_1 - i_1 - 1, k_2 - i_2 - 1\} \\ 0 & \text{otherwise.} \end{cases}
$$

$(\mathbf{r = n - 2})$ Suppose that there are $\lambda_{i_1} > \lambda_{i_2}$, $\mu_{j_1} > \mu_{j_2}$, $\nu_{k_1} > \nu_{k_2}$ such that $\lambda_{i_1+1} < \lambda_{i_1}$, $\lambda_{i_2+1} < \lambda_{i_2}$, $\mu_{j_1+1} < \mu_{j_1}$, $\mu_{j_2+1} < \mu_{j_2}$ and $\nu_{k_1} < \nu_{k_1-1}$, $\nu_{k_2} < \nu_{k_2-1}$ satisfying the following conditions;

$$
\begin{aligned}
(*) \quad \lambda_{i_1} + \lambda_{i_2} + \mu_{j_1} + \mu_{j_2} &\geq \nu_{k_1} + \nu_{k_2} + 2\nu_1 + 1\,, \\
\lambda_{i_1} + \mu_{j_1} &\geq \nu_1 + \nu_{k_1} + 1\,, \\
\lambda_{i_1} + \mu_{j_2} &\geq \nu_1 + \nu_{k_2} + 1\,, \\
\lambda_{i_2} + \mu_{j_1} &\geq \nu_1 + \nu_{k_2} + 1\,.
\end{aligned}
$$

a) If $i_1 + i_2 + j_1 + j_2 > k_1 + k_2 - 2$, then $c^{\nu}_{\lambda,\mu} = 0$.

b) If $i_1 + i_2 + j_1 + j_2 = k_1 + k_2 - 2$, then

$$
c^{\nu}_{\lambda,\mu} = \begin{cases} c^{\nu-(2^{k_1-1},1^{k_2-k_1})}_{\lambda-(2^{i_1},1^{i_2-i_1}),\,\mu-(2^{j_1},1^{j_2-j_1})} & \text{if } \; k_2 - i_2 - 1 \leq j_2 \leq k_2 - i_1 - 1, \\ & k_1 - i_2 - 1 \leq j_1 \\ & j_1 \leq \min\{k_1 - i_1 - 1, k_2 - i_2 - 1\}, \\ 0 & \text{otherwise.} \end{cases}
$$

# Examples

Note: The conjugated reduction formulae can be applied when we formally set $\lambda_0 = \mu_0 = \nu_1$ and $\nu_{n+1} = 0$ in each case.

- A primitive triple (no formula is applicable)

$$\nu = (9, 7, 6, 6, 3, 3, 2)$$

$$\lambda = (6, 5, 4, 3, 2, 1, 0)$$

$$\mu = (7, 3, 2, 1, 1, 1, 0)$$

- Small examples

$$c^{(5,4)}_{(4,2),(3,0)} \overset{\widetilde{n-1}^*}{=} c^{(3,2)}_{(2,0),(3,0)} \overset{2}{=} 1$$

$$c^{(5,3)}_{(4,0),(4,0)} \overset{2}{=} 1$$

$$
\begin{aligned}
& c^{(15,14,14,13,13,13,13,13,11,11,10,10,9,9)}_{(13,13,12,11,11,11,9,9,7,5,4,3,1,0),(9,8,7,6,6,5,5,5,4,2,2,0,0,0)} \\
\overset{2}{=} \quad & c^{(15,14,14,13,13,13,13,13,11,11,10,10)}_{(13,13,12,11,11,11,9,9,7,5,4,3),(7,6,6,5,5,5,4,2,2,0,0,0)} \\
\overset{\widetilde{n-1}^*}{=} \quad & c^{(12,11,11,10,10,10,10,10,8,8,7,7)}_{(10,10,9,8,8,8,6,6,4,2,1,0),(7,6,6,5,5,5,4,2,2,0,0,0)} \\
\overset{n-2}{=} \quad & c^{(11,10,10,10,10,10,8,8,7,7)}_{(10,10,9,8,8,6,4,2,1,0),(7,6,6,5,5,2,2,0,0,0)} \\
\overset{\widetilde{n-2}}{=} \quad & c^{(9,9,9,9,9,9,7,7,6,6)}_{(9,9,8,7,7,5,4,2,1,0),(6,5,5,4,4,2,2,0,0)} \\
\overset{n-1^*}{=} \quad & c^{(9,9,9,9,7,7,6,6)}_{(8,7,7,5,4,2,1,0),(6,5,5,4,4,2,2,0)} \\
\overset{\widetilde{n-2}}{=} \quad & c^{(7,7,7,7,5,5,4,4)}_{(6,5,5,5,4,2,1,0),(4,3,3,2,2,2,2,0)} \overset{2}{=} c^{(7,7,7,5,5,4)}_{(6,5,5,4,2,0),(4,3,2,2,2,0)} \\
\overset{\widetilde{n-2}}{=} \quad & c^{(5,5,5,4,4,3)}_{(5,4,4,3,2,0),(3,2,1,1,1,0)} \overset{n-1}{=} c^{(5,5,4,4,3)}_{(4,4,3,2,0),(3,2,1,1,1)} \\
\overset{n-2}{=} \quad & c^{(4,4,3)}_{(3,2,0),(3,2,1)} = \overset{n-1}{=} c^{(4,3)}_{(3,0),(3,1)} \overset{n-2}{=} c^{\varnothing}_{\varnothing,\varnothing} = 1
\end{aligned}
$$

# Applications

**Theorem [C, Moon 2011 ]** If $\lambda, \mu, \nu$ are partitions with $n$ *distinct* parts and $c^{\nu}_{\lambda\mu} = 1$, then we can reduce the triple $(\lambda, \mu, \nu)$ to $(\emptyset, \emptyset, \emptyset)$ using $r = 1$, 2 reductions.

Proof: Factorization + Fulton's Conjecture + Induction on $n$

**Conjecture** If $c^{\nu}_{\lambda\mu} = 1$, we can reduce the triple to $(\emptyset, \emptyset, \emptyset)$ using $r = 1, n-1, \widetilde{1}, \widetilde{n-1}$ reductions.

# Schur $P$-functions [Schur 1911]

For a strict partition $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell) \vdash k$,

$$Q_\lambda(x_1, x_2, \ldots, x_n) := \mathrm{Pf}(Q_{(\lambda_i,\lambda_j)}) \,,$$

where *Pfaffian* of a $2m \times 2m$ skew symmetric matrix $A = (a_{ij})$ is

$$\mathrm{Pf}(A) = \sum_{w \in S_{2m}} \varepsilon(w) \prod_{i=1}^{m} a_{w(2i-1)\, w(2i)} \,,$$

for $w(2i-1) < w(2i)$ and $w(1) < w(3) < \cdots < w(2m-3) < w(2m-1)$

$$Q_{(r,s)} = q_r q_s + 2 \sum_{i=1}^{s} (-1)^i q_{r+i} q_{s-i},$$

$$q_r(x_1, \ldots, x_n) = 2 \sum_{i=1}^{n} x_i^r \prod_{j \neq i} \frac{x_i + x_j}{x_i - x_j}$$

$$P_\lambda(x_1, x_2, \ldots, x_n) := 2^{-\ell} Q_\lambda(x_1, x_2, \ldots, x_n)$$

A specialized Hall-Littlewood function ($t = -1$)

$$P_\lambda(x_1, x_2, \ldots, x_n) = \frac{1}{(n-\ell)!} \sum_{w \in S_n} \prod_{i=1}^{n} x_{w(i)}^{\lambda_i} \prod_{i \le \ell, i<j} \frac{x_{w(i)} + x_{w(j)}}{x_{w(i)} - x_{w(j)}}$$

$$P_\lambda(x_1, x_2, \ldots, x_n) = \frac{\mathrm{Pf}\begin{pmatrix} \frac{x_i - x_j}{x_i + x_j} & x_i^{\lambda_j} \\ -x_j^{\lambda_i} & 0 \end{pmatrix}}{\mathrm{Pf}\left(\frac{x_i - x_j}{x_i + x_j}\right)}$$

$P_\lambda(x_1, \ldots, x_n)$ is a symmetric polynomial and $\{P_\lambda\}$ forms a basis of $\Gamma = \mathbb{C}[q_1, q_2, q_3, \ldots] \subset \Lambda$

# Combinatorial model for $P_\lambda$ (1)

Marked shifted semistandard tableaux of shape $\lambda = (5,3,2)$ on letters $\{1' < 1 < 2' < 2 < \cdots\}$:

$$\begin{array}{ccccc} 1 & 1 & 1 & 2' & 2 \\ & 2 & 2 & 5' & \\ & & 4 & 5' & \end{array}$$

$$P_\lambda = \sum_T x^T, \ T\text{: marked shifted semistandard tableau}$$

$$P_{(3,1)}(x_1, x_2) = x_1^3 x_2 \ + \ x_1^2 x_2^2 \ + \ x_1^2 x_2^2 \ + \ x_1 x_2^3$$

$$\begin{array}{ccc} 1 & 1 & 1 \\ & 2 & \end{array} \quad \begin{array}{ccc} 1 & 1 & 2' \\ & 2 & \end{array} \quad \begin{array}{ccc} 1 & 1 & 2 \\ & 2 & \end{array} \quad \begin{array}{ccc} 1 & 2' & 2 \\ & 2 & \end{array}$$

# Combinatorial model for $P_\lambda$ (2)

A word $w = w_1 w_2 \cdots w_m$ on the set of alphabets $\{1, 2, \ldots, n\}$ is a *hook word* if there is $1 \le m' \le m$ such that

$$w_1 > w_2 > \cdots > w_{m'} \le w_{m'+1} \le \cdots \le w_m .$$

A semistandard decomposition tableau (SSDT) $R$ of shape $\lambda$ is a filling of $\mathcal{S}(\lambda)$ such that

1. the word $R_i$ obtained by reading the $i$th row of $R$ from the left is a hook word of length $\lambda_i$, and

2. $R_i$ is a hook word of maximum length in $R_\ell R_{\ell-1} \cdots R_i$ for all $i = 1, \ldots, \ell - 1$.

$$P_\lambda = \sum_T x^T, \ T\text{: SSDT}$$

$$P_{(3,1)}(x_1, x_2) = x_1^3 x_2 \quad + \quad x_1^2 x_2^2 \quad + \quad x_1^2 x_2^2 \quad + \quad x_1 x_2^3$$

$$\begin{array}{ccc} 2 & 1 & 1 \\ & 1 & \end{array} \quad \begin{array}{ccc} 2 & 1 & 1 \\ & 2 & \end{array} \quad \begin{array}{ccc} 2 & 1 & 2 \\ & 1 & \end{array} \quad \begin{array}{ccc} 2 & 1 & 2 \\ & 2 & \end{array}$$

[Serrano 2010] There is a weight preserving bijection between the set of shifted semistandard tableaux and the set of SSDT's

$$\begin{array}{cccc} 1 & 2' & 3' & 3 \\ & 2 & 3' & 4 \\ & & 3 & \end{array} \quad \overset{\text{mixed insertion}}{\longleftarrow} \quad 3\,3\,2\,3\,4\,2\,1\,3 \quad \overset{\text{SK insertion}}{\longrightarrow} \quad \begin{array}{cccc} 4 & 2 & 1 & 3 \\ & 3 & 2 & 3 \\ & & 3 & \end{array}$$

# Lowest weight SSDT and highest weight SSDT

$$\lambda = (9, 8, 6, 4, 3)$$

$$\begin{array}{ccccccccc} 5 & 4 & 3 & 2 & 1 & | & 1 & 1 & 3 & 4 \\ & 5 & 4 & 3 & 2 & | & 2 & 2 & 4 & 5 \\ & & 5 & 4 & 3 & | & 3 & 3 & 5 & \\ & & & 5 & 4 & | & 4 & 4 & & \\ & & & & 5 & | & 5 & 5 & & \end{array} \qquad \begin{array}{cccccccccc} 5 & 4 & 3 & 2 & | & 1 & 1 & 1 & 1 & 1 \\ & 5 & 4 & 3 & | & 1 & 2 & 2 & 2 & 2 \\ & & 5 & 4 & | & 1 & 2 & 3 & 3 & \\ & & & 4 & | & 1 & 2 & 3 & & \\ & & & & | & 1 & 2 & 3 & & \end{array}$$

# Bender-Knuth type involutions: a proof of symmetry of $P_\lambda$

**Theorem [C 2013]** Lascoux-Schützenberger involutions defined on the set of words are Bender-Knuth type involution on SSDT's

$$\sigma_1 \begin{pmatrix} 6 & 5 & 4 & 2 & 1 & 1 & 3 \\ & 6 & 5 & 2 & 1 & 4 & \\ & & 5 & \color{red}{1} & 2 & 3 & \end{pmatrix} = \begin{matrix} 6 & 5 & 4 & 2 & 1 & 1 & 3 \\ & 6 & 5 & 2 & 1 & 4 & \\ & & 5 & \color{red}{2} & 2 & 3 & \end{matrix}$$

$$5\,\underline{1}\,\boxed{2}\,3\,6\,5\,\boxed{2}\,\boxed{1}\,4\,6\,5\,4\,\boxed{2}\,\boxed{1}\,\boxed{1}\,3 \rightarrow 5\,\underline{2}\,\boxed{2}\,3\,6\,5\,\boxed{2}\,\boxed{1}\,4\,6\,5\,4\,\boxed{2}\,\boxed{1}\,\boxed{1}\,3$$

# Related algebras

- $\Gamma$: subring of symmetric functions

$$P_\lambda(x)P_\mu(x) = \sum_\nu f_{\lambda,\mu}^{\nu} P_\nu(x)$$

- $H^*(OG(n+1, 2n+2))$: cohomology ring of orthogonal maximal isotropic Grassmannian

$$\tau_\lambda \tau_\mu = \sum_\nu f_{\lambda,\mu}^{\nu} \tau_\nu$$

- projective representations of $\mathfrak{S}_n$

## Shifted Littlewood-Richardson coefficients (1)

[Stembridge, 1989]

$$f^{\nu}_{\lambda,\mu} = \text{ number of LRS-tableaux}$$

When $\nu = (5,4,2,1)$, $\lambda = (3,1)$, $\mu = (4,3,1)$,

$$T = \begin{array}{ccccc} \cdot & \cdot & \cdot & 1' & 1 \\ & \cdot & 1 & 1 & 2' \\ & & 2 & 2 & \\ & & & 3 & \end{array} \quad \text{is an LRS tableau:}$$

- $T$ is a marked shifted semistandard tableau of shape $\nu/\lambda$ and content $\mu$

- Let $w\hat{w} = a_1 a_2 \cdots a_{2m}$ for $w = 1\,1'\,2'\,1\,1\,2\,2\,3$, $\hat{w} = 4'\,3'\,3'\,2'\,2'\,2\,1\,2'$. For each $a_i = k+1$ or $a_i = (k+1)'$, there are more $k$'s than $(k+1)$'s in $a_1 \cdots a_{i-1}$.

- The last occurrence of $k'$ precedes the last occurrence of $k$ in $w$.

# Shifted Littlewood-Richardson coefficients (2)

$$f^{\nu}_{\lambda,\mu} \quad = \quad \text{number of ??? SSDT ?}$$

Motivation:

Stembridge, *A concise proof of the Littlewood-Richarson rule*, Electron. J. Combin. 9 (2002)

$$P_\lambda(x_n, x_{n-1}, \ldots, x_1) = \frac{\mathrm{Pf}_\lambda(x_n, \ldots, x_1)}{\mathrm{Pf}_0(x_n, \ldots, x_1)}$$

$$= \quad \frac{1}{(n-\ell)!} \sum_{\pi \in S_n} \pi \left( x_n^{\lambda_1} x_{n-1}^{\lambda_2} \cdots x_{n-\ell+1}^{\lambda_\ell} \prod_{n-\ell+1 \leq j,\ i<j} \frac{x_j + x_i}{x_j - x_i} \right)$$

Let $D_n = \mathrm{Pf}_0(x_n, \ldots, x_1) = \prod_{1 \leq i < j \leq n} \frac{x_j - x_i}{x_j + x_i}$, then

$$D_n \cdot P_\lambda(x_n,\ldots,x_1)P_\mu(x_n,\ldots,x_1) \times (n-\ell)!$$

$$= \sum_{T\in\mathcal{Y}_n(\mu)} \left( \sum_{\pi\in S_n} \varepsilon(\pi)\ \pi\left( x_n^{\lambda_1}\cdots x_{n-\ell+1}^{\lambda_\ell}\ x^{\omega(T)} \prod_{1\le i<j\le n-\ell} \frac{x_j-x_i}{x_j+x_i} \right)\right)$$

$$= \sum_{\substack{R\in\mathcal{D}_n(\mu)\\ R \text{ is } \ell\text{-essential}}} \sum_{\pi\in S_n} \varepsilon(\pi)\pi\left( x^{r(\lambda)+\omega(R|_{n-\ell+1}^{n})} P_{sh(R|_1^{n-\ell})}(x_{n-\ell},\ldots,x_1) \prod_{1\le i<j\le n-\ell} \frac{x_j-x_i}{x_j+x_i}\right)$$

$$= \sum_R \frac{(n-\ell)!}{(n-\ell-k_R)!} \sum_{\pi\in S_n} \varepsilon(\pi)\pi\left( x^{r(\lambda)+\omega(R|_{n-\ell+1}^{n})} x_{n-\ell}^{\alpha_1^R}\cdots x_{n-\ell-k_R+1}^{\alpha_{k_R}^R} \prod_{1\le i<j\le n-\ell-k_R} \frac{x_j-x_i}{x_j+x_i}\right),$$

where $\alpha^R = (\alpha_1^R,\ldots,\alpha_{k_R}^R) = sh(R|_1^{n-\ell})$ and

an SSDT R is $\ell$-essential if $\omega(R|_1^{n-\ell}) = (\rho_{n-\ell},\ldots,\rho_1)$ where $\rho = sh(R|_1^{n-\ell})$.

When $n = 3$,

$R_2 = \begin{array}{ccc} 3 & 1 & 2 \\ & 2 & \end{array}$ is not 1-essential: $P_{mix}(2\,1\,2) = \begin{array}{ccc} 1 & 2' & 2 \end{array}$ is of shape $(3, 0)$ and is not the lowest weight tableau of shape $(3, 0)$.

$$D_n \cdot P_\lambda(x_n, \ldots, x_1) P_\mu(x_n, \ldots, x_1) = \sum_{R:\ell-\mathrm{essential}} D_n \cdot P_{\lambda + r(\omega(R))}(x_n, \ldots, x_1)$$

There are only seven 1-essential SSDT's among 24 SSDT's of shape $\mu = (3, 1)$, when $n = 3$:

| SSDT $R$ | $\begin{matrix} 2 & 1 & 2 \\ & 2 & \end{matrix}$ | $\begin{matrix} 3 & 2 & 2 \\ & 2 & \end{matrix}$ | $\begin{matrix} 3 & 2 & 1 \\ & 2 & \end{matrix}$ | $\begin{matrix} 2 & 1 & 3 \\ & 2 & \end{matrix}$ |
|---|---|---|---|---|
| $r(\omega(R))$ | $(0, 3, 1)$ | $(1, 3, 0)$ | $(1, 2, 1)$ | $(1, 2, 1)$ |
| $\lambda + r(\omega(R))$ | $(2, 3, 1)$ | $(3, 3, 0)$ | $(3, 2, 1)$ | $(3, 2, 1)$ |

| SSDT $R$ | $\begin{matrix} 3 & 2 & 3 \\ & 2 & \end{matrix}$ | $\begin{matrix} 3 & 2 & 2 \\ & 3 & \end{matrix}$ | $\begin{matrix} 3 & 2 & 3 \\ & 3 & \end{matrix}$ |
|---|---|---|---|
| $r(\omega(R))$ | $(2, 2, 0)$ | $(2, 2, 0)$ | $(3, 1, 0)$ |
| $\lambda + r(\omega(R))$ | $(4, 2, 0)$ | $(4, 2, 0)$ | $(5, 1, 0)$ |

$$
\begin{aligned}
&D_3 \cdot P_{(2,0,0)}(x_3,x_2,x_1)P_{(3,1,0)}(x_3,x_2,x_1)\\
&\quad = \quad D_3 \cdot P_{(2,3,1)}(x_3,x_2,x_1) + D_3 \cdot P_{(3,3,0)}(x_3,x_2,x_1) + D_3 \cdot P_{(3,2,1)}(x_3,x_2,x_1)\\
&\qquad\quad +D_3 \cdot P_{(3,2,1)}(x_3,x_2,x_1) + D_3 \cdot P_{(4,2,0)}(x_3,x_2,x_1)\\
&\qquad\quad +D_3 \cdot P_{(4,2,0)}(x_3,x_2,x_1) + D_3 \cdot P_{(5,1,0)}(x_3,x_2,x_1)
\end{aligned}
$$

An $\ell$-essential SSDT $R$ is $\lambda$-bad if $\mathrm{read}(R) = u_1u_2\cdots u_m$ is $\lambda$-bad:
There is $i$ such that $\lambda + r(\omega(u_1\ldots u_i))$ is not a strict partition.

Among seven 1-essential SSDT's of shape $\mu = (3,1)$,

$R_1 = \begin{matrix} 2 & 1 & 2 \\ & 2 & \end{matrix}$, $R_2 = \begin{matrix} 3 & 2 & 2 \\ & 2 & \end{matrix}$ and $R_3 = \begin{matrix} 2 & 1 & 3 \\ & 2 & \end{matrix}$ are $\lambda$-bad,

where $\lambda = (2)$:

$\mathrm{read}(R_1) = 2\,2\,1\,2$, $r(\omega(2\,2)) = (0,2,0)$ and $(2,0,0) + (0,2,0) \notin \mathcal{DP}$, Define a sign reversing involution on the set of $\lambda$-bad $\ell$-essential SSDT's:

For $u = u_1 u_2 \ldots u_m = \text{read}(R)$ let $i_0$ be the first $i$ such that $\lambda + r(\omega(u_1 \ldots u_i)) \notin \mathcal{DP}$, and $u_{i_0} = k$ then

$$R^{\diamond} = \sigma_k^{i_0} \sigma_{k-1} \cdots \sigma_{k-d_R} \cdots \sigma_{k-1} \sigma_k^{i_0}(R)$$

We have two relations

$D_3 \cdot P_{(2,3,1)}(x_3, x_2, x_1) = -D_3 \cdot P_{(3,2,1)}(x_3, x_2, x_1)$ and $D_3 \cdot P_{(3,3)}(x_3, x_2, x_1) = 0$. Therefore,

$$\begin{aligned} & D_3 \cdot P_{(2,0,0)}(x_3, x_2, x_1) P_{(3,1,0)}(x_3, x_2, x_1) \\ & = D_3 \cdot P_{(3,2,1)}(x_3, x_2, x_1) + 2D_3 \cdot P_{(4,2,0)}(x_3, x_2, x_1) + D_3 \cdot P_{(5,1,0)}(x_3, x_2, x_1) \end{aligned}$$

**Theorem** [C 2013]

$$P_\lambda(x_n,\dots,x_1)P_\mu(x_n,\dots,x_1) = \sum_R P_{\lambda+r(\omega(R))}(x_n,\dots,x_1),$$

where the sum runs over the $\lambda$-good $\ell$-essential SSDT's of shape $\mu$. Therefore

$f^\nu_{\lambda,\mu}$ = number of $\lambda$-good SSDT's of shape $\mu$ with weight $r(\nu-\lambda)$.

## Remark

- S.-J. Kang et.al proved that the set of SSDT*'s forms a crystal of the quantum queer superalgebra $\mathsf{U}_q(\mathfrak{q}(\mathfrak{n}))$ and obtain a similar result to the theorem on the decomposition of the product of two Schur $\mathsf{P}$-functions using crystal basis theory.

## Generalizations

- quasi-
- affine-
- - of Lie type (algebraic and geometric)
- $\mathsf{G/P} \rightsquigarrow \mathsf{G/B}$
- equivariant-
- quantum-
- K-theoretic

Thank you !!